Supplementary Materials for "A Sparse Interactive Model for Matrix Completion with Side Information"

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Here we present the proof of the three theoretical parts related to our work, including the exact recovery analysis, ϵ-recovery analysis and the convergence analysis. Algorithm details are followed from the theoretical part. The feature description table of the drug discovery dataset is displayed in the last section.

1 Exact Recovery Sampling Complexity

Theorem 1 *Let $\mu = \max(\mu_0, \mu_{\mathbf{XY}})$, $\sigma = \max(\|\mathbf{\Sigma_X}^{-1}\|_*, \|\mathbf{\Sigma_Y}^{-1}\|_*)$. μ_0, $\mu_{\mathbf{XY}}$ are calculated from $\mathbf{F}$. Denote $q_0 = \frac{1}{2}(1 + \log a - \log r), T_0 = \frac{128p}{3}\sigma\mu\max(\mu_1, \mu)r(a + b)\log N$ and $T_1 = \frac{8p}{3}\sigma^2\mu^2(ab + r^2)\log N$. Assume $T_1 \geq q_0 T_0$, $\mathbf{X}$ and $\mathbf{Y}$ are orthonormal. For any $p > 1$, with a probability at least $1 - 4(q_0 + 1)N^{-p+1} - 2q_0 N^{-p+2}$, $\mathbf{G}_0$ and $\mathbf{E}_0$ are the unique optimizer to the problem (3) in our formulation if*

$$|\Omega| \geq \frac{64p}{3}\sigma\mu\max(\mu_1, \mu)(1 + \log a - \log r)r(a + b)\log N$$

For proving Theorem 1, we introduce the Lemma 1 stating two deterministic conditions for $\mathbf{G}_0$ and $\mathbf{E}_0$ to be the unique minimizer of our problem. It can be proved in Lemma 2, Lemma 9 and Lemma 10 that under a high probability the assumption A1 and A2 hold. So let us first give Lemma 1 as below:

Lemma 1 *We assume that for any $\mathbf{M} \neq 0$, $\mathbf{M} \in \mathbb{R}^{m \times n}$ satisfying $R_\Omega(\mathbf{M}) = 0$ and $\mathbf{M} = P_{\mathbf{X}}\mathbf{M}P_{\mathbf{Y}}$, then we have*

$$\textbf{A1}\ \ \|P_T(\mathbf{M})\|_F \leq \zeta\|P_{T^\perp}(\mathbf{M})\|_F,$$

where

$$\zeta \leq \sqrt{\frac{a}{2r}}$$

And assume that there exists a matrix $\mathbf{H} \in \mathbb{R}^{m \times n}$ such that

$$\textbf{A2}\ \ R_\Omega(\mathbf{H}) = \mathbf{H}, \|P_T(\mathbf{H}) - \mathbf{U}\mathbf{V}^T\|_F \leq \sqrt{\frac{r}{2a}}, \|P_{T^\perp}(\mathbf{H})\| < \frac{1}{2}$$

Moreover we assume that there exists a constant C_0 such that

$$\textbf{A3}\ \ \|\mathbf{G}_0\|_1 = s < \frac{\lambda_G(\frac{1}{2} - \zeta\sqrt{\frac{r}{2a}})}{C\lambda_E},$$

where $C < C_0$, then $\mathbf{G}_0$ and $\mathbf{E}_0$ are the unique minimizer to our optimization problem.

Proof. Assuming the solution is not unique, there exists another solution $\mathbf{G}_0 + \Delta\mathbf{G}$ and $\mathbf{E}_0 + \Delta\mathbf{E}$ with $\Delta\mathbf{G}, \Delta\mathbf{E} \neq 0$. Basically our aim is to prove the contradiction that $\|\mathbf{G}_0 + \Delta\mathbf{G}\|_1 + \|\mathbf{E}_0 + \Delta\mathbf{E}\|_* \geq \|\mathbf{G}_0\|_1 + \|\mathbf{E}_0\|_*$.

In order to prove the contradiction, we illustrate several useful facts below:

(1) $R_\Omega(\mathbf{X}^T(\mathbf{G}_0 + \Delta\mathbf{G})\mathbf{Y}) = R_\Omega(\mathbf{X}^T\mathbf{G}_0\mathbf{Y})$ and $\mathbf{X}^T(\mathbf{G}_0 + \Delta\mathbf{G})\mathbf{Y} = P_\mathbf{X}(\mathbf{X}^T(\mathbf{G}_0 + \Delta\mathbf{G})\mathbf{Y})P_\mathbf{Y}$, as $\mathbf{G}_0 + \Delta\mathbf{G}$ minimizes the original problem.

(2) $\mathbf{X}^T\Delta\mathbf{G}\mathbf{Y} \neq 0$, since $\mathbf{X}\mathbf{X}^T\Delta\mathbf{G}\mathbf{Y}\mathbf{Y}^T \neq 0$ for $\mathbf{X}$ and $\mathbf{Y}$ are full row rank.

(3) $\mathbf{X}^T\Delta\mathbf{G}\mathbf{Y} = P_\mathbf{X}(\mathbf{X}^T\Delta\mathbf{G}\mathbf{Y})P_\mathbf{Y}$, $R_\Omega(\mathbf{X}\Delta\mathbf{G}\mathbf{Y}) = 0$.

(4) $\|P_T(\mathbf{X}^T\Delta\mathbf{G}\mathbf{Y})\|_F \leq \zeta\|P_T(\mathbf{X}^T\Delta\mathbf{G}\mathbf{Y})\|_F \leq \zeta\|P_{T^\perp}(\mathbf{X}^T\Delta\mathbf{G}\mathbf{Y})\|_*$ since $\mathbf{X}^T\mathbf{G}_0\mathbf{Y} \neq 0$ with Condition A1.

(5) $\mathbf{U}_\perp$ and $\mathbf{V}_\perp$ are the left and right singular vectors of $P_T(\mathbf{X}^T\Delta\mathbf{G}\mathbf{Y})$, while $\mathbf{U}^T\mathbf{U}_\perp = 0$ and $\mathbf{V}^T\mathbf{V}_\perp = 0$.

Denote $\lambda = \lambda_G/\lambda_E$ for simplicity, then we can obtain that

$$\|\mathbf{E}_0 + \Delta\mathbf{E}\|_* + \lambda\|\mathbf{G}_0 + \Delta\mathbf{G}\|_1$$
$$=\|\mathbf{X}^T(\mathbf{G}_0 + \Delta\mathbf{G})\mathbf{Y}\|_* + \lambda\|\mathbf{G}_0 + \Delta\mathbf{G}\|_1$$
$$=\|\mathbf{X}^T(\mathbf{G}_0 + \Delta\mathbf{G})\mathbf{Y}\|_*\|\mathbf{U}\mathbf{V}^T + \mathbf{U}_\perp\mathbf{V}_\perp^T\| + \lambda\|\mathbf{G}_0 + \Delta\mathbf{G}\|_1$$

since the $\|\mathbf{U}\mathbf{V}^T + \mathbf{U}_\perp\mathbf{V}_\perp^T\| = 1$, where the norm is the operator norm. Then

$$\|\mathbf{X}^T(\mathbf{G}_0 + \Delta\mathbf{G})\mathbf{Y}\|_*\|\mathbf{U}\mathbf{V}^T + \mathbf{U}_\perp\mathbf{V}_\perp^T\| + \lambda\|\mathbf{G}_0 + \Delta\mathbf{G}\|_1$$
$$\geq \left\langle \mathbf{X}^T(\mathbf{G}_0 + \Delta\mathbf{G})\mathbf{Y}, \mathbf{U}\mathbf{V}^T + \mathbf{U}_\perp\mathbf{V}_\perp^T \right\rangle + \lambda\|\mathbf{G}_0 + \Delta\mathbf{G}\|_1$$
$$= \left\langle \mathbf{X}^T\mathbf{G}_0\mathbf{Y}, \mathbf{U}\mathbf{V}^T \right\rangle + \left\langle \mathbf{X}^T\mathbf{G}_0\mathbf{Y}, \mathbf{U}_\perp\mathbf{V}_\perp^T \right\rangle +$$
$$\left\langle \mathbf{X}^T\Delta\mathbf{G}\mathbf{Y}, \mathbf{U}\mathbf{V}^T + \mathbf{U}_\perp\mathbf{V}_\perp^T \right\rangle + \lambda\|\mathbf{G}_0 + \Delta\mathbf{G}\|_1$$
$$= \|\mathbf{E}_0\|_* + \left\langle \mathbf{X}^T\Delta\mathbf{G}\mathbf{Y}, \mathbf{U}\mathbf{V}^T + \mathbf{U}_\perp\mathbf{V}_\perp^T - \mathbf{H} \right\rangle + \lambda\|\mathbf{G}_0 + \Delta\mathbf{G}\|_1$$

This is obtained from the assumption A2 and matrix norm inequality. One can obtain the derivation by the norm inequality as follows,

$$\left\langle \mathbf{X}^T\Delta\mathbf{G}\mathbf{Y}, \mathbf{U}\mathbf{V}^T + \mathbf{U}_\perp\mathbf{V}_\perp^T - \mathbf{H} \right\rangle + \lambda\|\mathbf{G}_0 + \Delta\mathbf{G}\|_1$$
$$= \left\langle P_T(\mathbf{X}^T\Delta\mathbf{G}\mathbf{Y}), \mathbf{U}\mathbf{V}^T - P_T(\mathbf{H}) \right\rangle + \left\langle P_{T^\perp}(\mathbf{X}^T\Delta\mathbf{G}\mathbf{Y}), \mathbf{U}_\perp\mathbf{V}_\perp^T - P_{T^\perp}(\mathbf{H}) \right\rangle + \lambda\|\mathbf{G}_0 + \Delta\mathbf{G}\|_1$$
$$\geq \|P_{T^\perp}(\mathbf{X}^T\Delta\mathbf{G}\mathbf{Y})\|_* - \|P_T(\mathbf{X}^T\Delta\mathbf{G}\mathbf{Y})\|_F\|\mathbf{U}\mathbf{V}^T - P_T(\mathbf{H})\|_F$$
$$- \|P_{T^\perp}(\mathbf{H})\|_F\|P_{T^\perp}(\mathbf{X}^T\Delta\mathbf{G}\mathbf{Y})\|_* + \lambda\|\mathbf{G}_0 + \Delta\mathbf{G}\|_1$$

$$(1)$$

using the assumption A1 and A2 then organizing the terms, we could get

$$\|\mathbf{E}_0 + \Delta\mathbf{E}\|_* + \lambda\|\mathbf{G}_0 + \Delta\mathbf{G}\|_1$$
$$> \|\mathbf{E}_0\|_* + \|P_{T^\perp}(\mathbf{X}^T\Delta\mathbf{G}\mathbf{Y})\|_*(\frac{1}{2} - \zeta\sqrt{\frac{r}{2a}}) + \lambda\|\mathbf{G}_0 + \Delta\mathbf{G}\|_1$$
$$\geq \|\mathbf{E}_0\|_* + \|P_{T^\perp}(\mathbf{X}^T\Delta\mathbf{G}\mathbf{Y})\|_*(\frac{1}{2} - \zeta\sqrt{\frac{r}{2a}}) + \lambda\|\mathbf{G}_0\|_1 - \lambda s \qquad (2)$$
$$\geq \|\mathbf{E}_0\|_* + \lambda\|\mathbf{G}_0\|_1 + \|P_{T^\perp}(\mathbf{X}^T\Delta\mathbf{G}\mathbf{Y})\|_*(\frac{1}{2} - \zeta\sqrt{\frac{r}{2a}}) - \lambda s$$

Since $\|P_{T^\perp}(\mathbf{X}^T\Delta\mathbf{G}\mathbf{Y})\|_*(\frac{1}{2} - \zeta\sqrt{\frac{r}{2a}}) \geq 0$ which is implied from assumption A2, by observing Eq. (2), once (i) $\lambda(\|\mathbf{G}_0 + \Delta\mathbf{G}\|_1) \geq \lambda s \geq \lambda\|\mathbf{G}_0\|_1$, or (ii) $\|P_{T^\perp}(\mathbf{X}^T\Delta\mathbf{G}\mathbf{Y})\|_*(\frac{1}{2} - \zeta\sqrt{\frac{r}{2a}}) - \lambda s \geq 0$ is proved, the result could lead to a contradiction. We prove it by separating the problem into two cases.

If the case (i) holds, we can directly obtain from Eq. (2) that $\|\mathbf{E}_0 + \Delta\mathbf{E}\|_* + \lambda\|\mathbf{G}_0 + \Delta\mathbf{G}\|_1 \geq \|\mathbf{E}_0\|_* + \lambda\|\mathbf{G}_0\|_1$.

In the contrary case of case (i), let us assume $\lambda(\|\mathbf{G}_0 + \Delta\mathbf{G}\|_1) < \lambda\|\mathbf{G}_0\|_1$. First consider the case if the possible minimizers $\mathbf{G}$'s exist in the small ε-ball $B_\epsilon(\mathbf{G}_0)$ as a continuous neighbour of $\mathbf{G}_0$, such that $\varepsilon < \min_{i,j}|\mathbf{G}_{ij}|$. Then for each $\mathbf{G}_s = \{\mathbf{G}_0 + \Delta\mathbf{G} \in B_\epsilon(\mathbf{G}_0)\}$, it satisfies $\|\mathbf{G}_s\|_1 \geq \|\mathbf{G}_0\|_1 - ab\varepsilon$. Hence $\lambda(\|\mathbf{G}_0 + \Delta\mathbf{G}\|_1) \geq \lambda s - ab\lambda\varepsilon$. Since ε is arbitrary, $\lambda(\|\mathbf{G}_0 + \Delta\mathbf{G}\|_1) \geq \lambda s$. Therefore, in this case the condition (i) is satisfied, which leads to the contradiction.

Otherwise, consider the minimizers $\mathbf{G}$'s exist outside of the ε-ball $B_\epsilon(\mathbf{G}_0)$, which means $\mathbf{G}_0$ is an isolated minimizer. Let us assume that there exists a constant C' such that for all $\Delta\mathbf{G}$,

$$\|P_{T^\perp}(\mathbf{X}^T\Delta\mathbf{G}\mathbf{Y})\|_F \geq C' > C > 0.$$

Here from the assumption A3 we can derive

$$\|P_{T^\perp}(\mathbf{X}^T\Delta\mathbf{G}\mathbf{Y})\|_*(\frac{1}{2} - \zeta\sqrt{\frac{r}{2a}}) - \lambda s$$

$$\geq \|P_{T^\perp}(\mathbf{X}^T\Delta\mathbf{G}\mathbf{Y})\|_F(\frac{1}{2} - \zeta\sqrt{\frac{r}{2a}}) - \lambda s$$

$$\geq C(\frac{1}{2} - \zeta\sqrt{\frac{r}{2a}}) - \lambda s$$

$$\geq 0$$

Thus the condition (ii) is satisfied.

Suppose there is no such a constant C' satisfying the above condition. This implies that there exists infinite minimizers and we can obtain a sub-sequence $\{\Delta\mathbf{G}_{t_k}\}_{k=1}^\infty$ satisfying

$$\lim_{k=\infty}\|P_{T^\perp}(\mathbf{X}^T\Delta\mathbf{G}_{t_k}\mathbf{Y})\|_F = 0 \tag{3}$$

Due to the nuclear norm inequality and the fact (4) we further infer that

$$0 \leq \lim_{k=\infty}\|P_T(\mathbf{X}^T\Delta\mathbf{G}_{t_k}\mathbf{Y})\|_F \leq \lim_{k=\infty}\|P_{T^\perp}(\mathbf{X}^T\Delta\mathbf{G}_{t_k}\mathbf{Y})\|_F = 0 \tag{4}$$

Combining Eq.(3) and Eq.(4) we have

$$\lim_{k=\infty}\|\mathbf{X}^T\Delta\mathbf{G}_{t_k}\mathbf{Y}\|_F = 0 \tag{5}$$

Eq.(5) implies that the infinite sequence $\{\mathbf{G} + \Delta\mathbf{G}_t\}_{t=N}^\infty \subset B_\epsilon(\mathbf{G}_0)$, which is contradicted to the fact that no minimizers $\mathbf{G}$'s exist within the ε-ball $B_\epsilon(\mathbf{G}_0)$. Therefore, the above all clarify the truth that the $\mathbf{E}_0 + \Delta\mathbf{E}$ and $\mathbf{G}_0 + \Delta\mathbf{G}$ are not the minimizer for our optimization problem. ∎

1.1 A1 holds with high probability

In this subsection we prove that Lemma 2 holds with some certain probability. Lemma 2 roots from combining the results from Lemma 5 and Lemma 6, which upper-bounds $\|P_T(\mathbf{M})\|_F$ and lower-bounds $\|P_{T^\perp}(\mathbf{M})\|_F$ and clarify the inequality between them. Lemma 3 and 4 are cited from [7] to facilitate the proof.

Let's first illustrate Lemma 2 as below.

Lemma 2 *With a certain probability at least $1 - 4N^{-p+1}$, for any $\mathbf{M} \neq 0$, $\mathbf{M} \in \mathbb{R}^{m\times n}$ satisfying $R_\Omega(\mathbf{M}) = 0$ and $\mathbf{M} = P_\mathbf{X}\mathbf{M}P_\mathbf{Y}$ we have*

$$\|P_T(\mathbf{M})\|_F \leq \zeta\|P_{T^\perp}(\mathbf{M})\|_F,$$

where ζ is the same as in Lemma 1, if $T_0 \leq |\mathbf{\Omega}| \leq T_1$.

 Proof. Since $R_\Omega(\mathbf{M}) = 0$ and $\mathbf{M} = P_\mathbf{X}\mathbf{M}P_\mathbf{Y}$, we have $R_\Omega P_T(\mathbf{M}) = -R_\Omega P_{T^\perp}(\mathbf{M})$. Then we could attain

$$\frac{mn}{|\mathbf{\Omega}|}\langle\mathbf{M}, P_T R_\Omega P_T(\mathbf{M})\rangle = \frac{mn}{|\mathbf{\Omega}|}\langle\mathbf{M}, P_{T^\perp} R_\Omega P_{T^\perp}(\mathbf{M})\rangle$$

First, according to Lemma 5 and Lemma 6, with a probability at least $1 - 4N^{-p+1}$, we have

$$
\begin{aligned}
\frac{1}{2}\|P_T(\mathbf{M})\|_F^2 &\leq \frac{mn}{|\mathbf{\Omega}|}\langle \mathbf{M}, P_T R_\Omega P_T(\mathbf{M})\rangle \\
&\leq \frac{16\sigma^2 p\mu^2(ab+r^2)\log N}{3|\mathbf{\Omega}|}\|P_{T^\perp}(\mathbf{M})\|_F^2 \\
&\leq \frac{16\sigma^2 p\mu^2(ab+r^2)\log N}{3T_0}\|P_{T^\perp}(\mathbf{M})\|_F^2 \\
&= \frac{1}{2}\|P_{T^\perp}(\mathbf{M})\|_F^2
\end{aligned}
$$

finally we have

$$
\frac{1}{2}\|P_T(\mathbf{M})\|_F \leq \frac{1}{\sqrt{2}}\|P_{T^\perp}(\mathbf{M})\|_F
$$

while $r \leq a$, we have $\frac{1}{\sqrt{2}} \leq \sqrt{\frac{a}{2r}}$, so the lemma proved. ∎

Before we prove Lemma 5 and Lemma 6, which we which upper-bounds $\|P_T(\mathbf{M})\|_F$ and lower-bounds $\|P_{T^\perp}(\mathbf{M})\|_F$ on certain conditions, we first need to illustrate Lemma 3, Lemma 4 derived from the Bernstein Inequality [5].

Lemma 3 *Let $\mathbf{X}_1, ..., \mathbf{X}_L$ be independent zero-mean random matrices of dimension $d_1 \times d_2$. Suppose $\rho_k^2 \geq \max\{\|\mathbb{E}[\mathbf{X}_k\mathbf{X}_k^T]\|, \|\mathbf{X}_k^T\mathbf{X}_k\|\}$ and $\|\mathbf{X}_k\| \leq M$ almost surely form all k. If we assume*

$$
M^2 \log\frac{d_1+d_2}{\xi} \leq \frac{3}{8}\sum \rho_k^2,
$$

then with a certain probability at least $1 - \xi$, we have,

$$
\|\sum_{k=1}^{L}\mathbf{X}_k\| \leq \sqrt{\frac{8}{3}\ln\frac{d_1+d_2}{\xi}\sum_{k=1}^{L}\rho_k^2}.
$$

We can also give Lemma 4 which can be derived from Lemma 3 as below;

Lemma 4 *Let $\mathbf{X}_1, ..., \mathbf{X}_L$ be independent zero-mean random matrices of dimension $d_1 \times d_2$. Suppose $\rho_k^2 \geq \max\{\|\mathbb{E}[\mathbf{X}_k\mathbf{X}_k^T]\|, \|\mathbf{X}_k^T\mathbf{X}_k\|\}$ and $\|\mathbf{X}_k\| \leq M$ almost surely form all k. If we assume*

$$
M^2 \log\frac{d_1+d_2}{\xi} \leq \frac{3}{8}\sum \rho_k^2,
$$

then with a certain probability at least $1 - \xi$, we have,

$$
\|\sum_{k=1}^{L}\mathbf{X}_k\| \leq \frac{8}{3}M\log\frac{d_1+d_2}{\xi}.
$$

next we will bound $\|P_T - \frac{mn}{|\Omega|}P_T R_\Omega P_T\|$ by using Lemma 3 and Lemma 4.

Lemma 5 *With a certain probability at least $1 - 2N^{-p+1}$, we have*

$$
\left\|P_T - \frac{mn}{|\mathbf{\Omega}|}P_T R_\Omega P_T\right\| \leq \sqrt{\frac{8p\mu^2 r(a+b)\log N}{3|\mathbf{\Omega}|}}
$$

if $|\Omega| \geq \frac{8p}{3}\mu^2 r(a+b)\log N$ and therefore, for any $\mathbf{M} \in \mathbb{R}^{m \times n}$,

$$
\frac{mn}{|\mathbf{\Omega}|}\langle \mathbf{M}, P_T R_\Omega P_T(\mathbf{M})\rangle \geq \frac{1}{2}\|P_T(\mathbf{M})\|_F^2
$$

if $|\mathbf{\Omega}| \geq T_0$.

Proof. For any $\mathbf{M} \in \mathbb{R}^{m \times n}$, we have

$$P_T R_\Omega P_T(\mathbf{M}) = \sum_{(i,j) \in \Omega} \left\langle P_T(\mathbf{M}), \mathbf{e}_i \mathbf{e}_j^T \right\rangle P_T(\mathbf{e}_i \mathbf{e}_j^T) = \sum_{(i,j) \in \Omega} \left\langle \mathbf{M}, P_T(\mathbf{e}_i \mathbf{e}_j^T) \right\rangle P_T(\mathbf{e}_i \mathbf{e}_j^T).$$

For any $i \in [m]$ and $j \in [n]$, define linear operator $T_{i,j}$ as

$$T_{i,j}(\mathbf{M}) = \left\langle \mathbf{M}, P_T(\mathbf{e}_i \mathbf{e}_j^T) \right\rangle P_T(\mathbf{e}_i \mathbf{e}_j^T) = P_T R_{(i,j)} P_T(\mathbf{M}),$$

where $R_{(i,j)}(\mathbf{M}) = \mathbf{e}_i \mathbf{e}_j^T \mathbf{M}_{i,j}$. So that

$$P_T R_\Omega P_T(\mathbf{M}) = \sum_{(i,j) \in \Omega} P_T R_{(i,j)} P_T(\mathbf{M}) = \sum_{(i,j) \in \Omega} T_{i,j}(\mathbf{M}).$$

To implement Lemma 3, we need to give M and the corresponding ρ^2. Since $\|P_T - \frac{mn}{|\mathbf{\Omega}|} P_T R_\Omega P_T\|$ can be viewed as the spectral norm of $|\mathbf{\Omega}|$ independent zero-mean random variables $\frac{1}{|\mathbf{\Omega}|} P_T - \frac{mn}{|\mathbf{\Omega}|} T_{i,j}$, then we have

$$\|\frac{1}{|\mathbf{\Omega}|} P_T - \frac{mn}{|\mathbf{\Omega}|} T_{i,j}\| \leq \max\{\|\frac{1}{|\mathbf{\Omega}|} P_T\|, \|\frac{mn}{|\mathbf{\Omega}|} T_{i,j}\|\}$$

$$= \max\{\|\frac{1}{|\mathbf{\Omega}|} P_T\|, \frac{mn}{|\mathbf{\Omega}|} \arg\max_{\|\mathbf{M}\|_F = 1} \|\left\langle \mathbf{M}, P_T(\mathbf{e}_i \mathbf{e}_j^T) \right\rangle P_T(\mathbf{e}_i \mathbf{e}_j^T)\|_F\}$$

$$= \max\{\|\frac{1}{|\mathbf{\Omega}|} P_T\|, \frac{mn}{|\mathbf{\Omega}|} \arg\max_{\|\mathbf{M}\|_F = 1} \left\langle \mathbf{M}, P_T(\mathbf{e}_i \mathbf{e}_j^T) \right\rangle \|P_T(\mathbf{e}_i \mathbf{e}_j^T)\|_F\}$$

$$= \max\{\|\frac{1}{|\mathbf{\Omega}|} P_T\|, \frac{mn}{|\mathbf{\Omega}|} \|P_T(\mathbf{e}_i \mathbf{e}_j^T)\|_F\}$$

To bound $\|P_T(\mathbf{e}_i \mathbf{e}_j^T)\|_F$, we get

$$\|P_T(\mathbf{e}_i \mathbf{e}_j^T)\|_F = \left\langle P_T(\mathbf{e}_i \mathbf{e}_j^T), \mathbf{e}_i \mathbf{e}_j^T \right\rangle$$

$$= \left\langle P_\mathbf{X}(\mathbf{e}_i \mathbf{e}_j^T) P_\mathbf{V}, \mathbf{e}_i \mathbf{e}_j^T \right\rangle + \left\langle P_\mathbf{U}(\mathbf{e}_i \mathbf{e}_j^T) P_\mathbf{Y}, \mathbf{e}_i \mathbf{e}_j^T \right\rangle - \left\langle P_\mathbf{U}(\mathbf{e}_i \mathbf{e}_j^T) P_\mathbf{V}, \mathbf{e}_i \mathbf{e}_j^T \right\rangle$$

$$= \|P_\mathbf{X}(\mathbf{e}_i \mathbf{e}_j^T) P_\mathbf{V}\|_F + \|P_\mathbf{U}(\mathbf{e}_i \mathbf{e}_j^T) P_\mathbf{Y}\|_F - \|P_\mathbf{U}(\mathbf{e}_i \mathbf{e}_j^T) P_\mathbf{V}\|_F$$

$$\leq \|P_\mathbf{X} \mathbf{e}_i\|_F \|P_\mathbf{V} \mathbf{e}_j\|_F + \|P_\mathbf{U} \mathbf{e}_i\|_F \|P_\mathbf{Y} \mathbf{e}_j\|_F$$

$$\leq \|\mathbf{X}^T \mathbf{V}_\mathbf{X} \mathbf{\Sigma}_\mathbf{X}^{-2} \mathbf{V}_\mathbf{X}^T\|_F \frac{a\mu_\mathbf{XY}}{m} \frac{r\mu_0}{n} + \|\mathbf{Y}^T \mathbf{V}_\mathbf{Y} \mathbf{\Sigma}_\mathbf{Y}^{-2} \mathbf{V}_\mathbf{Y}^T\|_F \frac{r\mu_0}{m} \frac{b\mu_\mathbf{XY}}{n}$$

$$\leq \|\mathbf{\Sigma}_\mathbf{X}^{-1}\|_* \frac{a\mu_\mathbf{XY}}{m} \frac{r\mu_0}{n} + \|\mathbf{\Sigma}_\mathbf{Y}^{-1}\|_* \frac{r\mu_0}{m} \frac{b\mu_\mathbf{XY}}{n}$$

$$\leq \sigma \frac{r\mu_0 \mu_\mathbf{XY}(a+b)}{mn} \leq \frac{\sigma r \mu^2(a+b)}{mn}.$$

Therefore

$$\|\frac{1}{|\mathbf{\Omega}|} P_T - \frac{mn}{|\mathbf{\Omega}|} T_{i,j}\| \leq \max\{\|\frac{1}{|\mathbf{\Omega}|} P_T\|, \frac{mn}{|\mathbf{\Omega}|} \|P_T(\mathbf{e}_i \mathbf{e}_j^T)\|_F\}$$

$$\leq \max\{\|\frac{1}{|\mathbf{\Omega}|} P_T\|, \frac{\sigma r\mu^2(a+b)}{mn}\} = \max\{\frac{1}{|\mathbf{\Omega}|}, \frac{\sigma r\mu^2(a+b)}{mn}\}$$

$$= \max\{\frac{1}{|\mathbf{\Omega}|}, \frac{\sigma r\mu^2(a+b)}{mn}\} = \frac{\sigma r\mu^2(a+b)}{mn} = M$$

5

Since that $\frac{1}{|\boldsymbol{\Omega}|}\mathbb{E}[P_T R_\Omega P_T(\mathbf{M})] = \frac{1}{mn}P_T(\mathbf{M})$, the corresponding $\rho_{i,j}^2$ can be calculated as

$$
\begin{aligned}
\rho_{i,j}^2 &= \|\mathbb{E}[(\frac{1}{|\boldsymbol{\Omega}|}P_T - \frac{mn}{|\boldsymbol{\Omega}|}T_{i,j})^T(\frac{1}{|\boldsymbol{\Omega}|}P_T - \frac{mn}{|\boldsymbol{\Omega}|}T_{i,j})]\| \\
&= \|\mathbb{E}[\frac{1}{|\boldsymbol{\Omega}|^2}P_T P_T + \frac{m^2 n^2}{|\boldsymbol{\Omega}|^2}T_{i,j}T_{i,j} - \frac{2mn}{|\boldsymbol{\Omega}|^2}P_T T_{i,j}]\| \\
&= \|\frac{1}{|\boldsymbol{\Omega}|^2}P_T + \frac{m^2 n^2}{|\boldsymbol{\Omega}|^2}\mathbb{E}[T_{i,j}T_{i,j}] - \frac{2mn}{|\boldsymbol{\Omega}|^2}P_T \mathbb{E}[T_{i,j}]\| \\
&= \|\frac{1}{|\boldsymbol{\Omega}|^2}P_T + \frac{m^2 n^2}{|\boldsymbol{\Omega}|^2}\mathbb{E}[T_{i,j}T_{i,j}] - \frac{2mn}{|\boldsymbol{\Omega}|^2}P_T \frac{1}{mn}P_T\| \\
&= \|\frac{m^2 n^2}{|\boldsymbol{\Omega}|^2}\mathbb{E}[T_{i,j}T_{i,j}] - \frac{1}{|\boldsymbol{\Omega}|^2}P_T\| \le \max\{\frac{m^2 n^2}{|\boldsymbol{\Omega}|^2}\mathbb{E}[T_{i,j}T_{i,j}], \frac{1}{|\boldsymbol{\Omega}|^2}P_T\} \\
&\le \max\{\frac{m^2 n^2}{|\boldsymbol{\Omega}|^2}\mathbb{E}[\|P_T(\mathbf{e}_i \mathbf{e}_j^T)\|_F \|T_{i,j}\|], \frac{1}{|\boldsymbol{\Omega}|^2}\} \\
&\le \max\{\frac{m^2 n^2}{|\boldsymbol{\Omega}|^2}\frac{\sigma r \mu^2(a+b)}{mn}\frac{1}{mn}\|P_T\|], \frac{1}{|\boldsymbol{\Omega}|^2}\} \\
&= \frac{\sigma r \mu^2(a+b)}{|\boldsymbol{\Omega}|^2}
\end{aligned}
$$

By Lemma 4, let $M = \frac{\sigma r \mu^2(a+b)}{mn}$ and $\rho^2 = \frac{\sigma r \mu^2(a+b)}{|\boldsymbol{\Omega}|^2}$, we conclude with a certain probability $1 - 2N^{-p+1}$,

$$
\|P_T - \frac{mn}{|\boldsymbol{\Omega}|}P_T R_\Omega P_T\| \le \sqrt{\frac{8}{3}\log\frac{m+n}{2N^{-p+1}}\frac{\sigma r \mu^2(a+b)}{|\boldsymbol{\Omega}|}} \le \sqrt{\frac{8\sigma p r \mu^2(a+b)\log N}{3|\boldsymbol{\Omega}|}}
$$

which also should satisfy the condition that

$$
\frac{\sigma^2 r^2 \mu^4 (a+b)^2}{|\boldsymbol{\Omega}|^2}\log\frac{m+n}{2N^{-p+1}} \le \frac{3}{8}\frac{\sigma r \mu^2(a+b)}{|\boldsymbol{\Omega}|}
$$

which means

$$
|\boldsymbol{\Omega}| \ge \frac{8\sigma p r \mu^2(a+b)\log N}{3}.
$$

Moreover, if $|\boldsymbol{\Omega}| \ge T_0 \ge \frac{32\sigma p r \mu^2(a+b)\log N}{3}$, then

$$
\left\|P_T - \frac{mn}{|\boldsymbol{\Omega}|}P_T R_\Omega P_T\right\| \le \sqrt{\frac{8\sigma p \mu^2 r(a+b)\log \tau}{3|\boldsymbol{\Omega}|}} \le \frac{1}{2},
$$

By utilizing the property of matrix norm, we have

$$
\left\langle \mathbf{M}, P_T(\mathbf{M}) - \frac{mn}{|\boldsymbol{\Omega}|}P_T R_\Omega P_T(\mathbf{M})\right\rangle \le \frac{1}{2}\|P_T(\mathbf{M})\|_F^2
$$

So that

$$
\langle \mathbf{M}, P_T(\mathbf{M})\rangle - \frac{1}{2}\|P_T(\mathbf{M})\|_F \le \left\langle \mathbf{M}, \frac{mn}{|\boldsymbol{\Omega}|}P_T R_\Omega P_T(\mathbf{M})\right\rangle,
$$

which we can easily derive

$$
\frac{1}{2}\|P_T(\mathbf{M})\|_F^2 \le \frac{mn}{|\boldsymbol{\Omega}|}P_T R_\Omega P_T(\mathbf{M}).
$$

∎

Following the similarly outline of the proof as Lemma 5, we can prove the following Lemma 6.

Lemma 6 *With a certain probability at least $1 - 2N^{-p+1}$, we have*

$$\left\| P_{T^\perp} - \frac{mn}{|\mathbf{\Omega}|} P_{T^\perp} R_\Omega P_{T^\perp} \right\| \leq \frac{8\sigma^2 p\mu^2(ab + r^2)\log N}{3|\mathbf{\Omega}|}$$

if $|\Omega| \geq T_0 = \frac{8}{3}\sigma\mu^2 pr(a + b)\log N$ and therefore, for any $\mathbf{M} \in \mathbb{R}^{m \times n}$,

$$\frac{mn}{|\mathbf{\Omega}|} \langle \mathbf{M}, P_{T^\perp} R_\Omega P_{T^\perp}(\mathbf{M}) \rangle \leq \frac{16\sigma^2 p\mu^2(ab + r^2)\log N}{3|\mathbf{\Omega}|} \| P_{T^\perp}(\mathbf{M}) \|_F^2$$

Then based on Lemma 5 and 6, we can prove that **A1** holds with a certain high probability.

1.2 A2 holds with high probability

In this subsection we aim to investigate the condition when **A2** holds with the high probability. Like the similar approach we propose above, we also need to bound the following two terms $\frac{mn}{|\mathbf{\Omega}|}\|P_{T^\perp} R_\Omega P_T(\mathbf{H})\|_F$ and $\|P_T(\mathbf{H}) - \frac{mn}{|\mathbf{\Omega}|} P_{T^\perp} R_\Omega P_T(\mathbf{H})\|_\infty$ in Lemma 7 and 8 respectively where $\|\cdot\|_\infty$ is the maximum entry of a matrix.

Lemma 7 *For a fixed $\mathbf{H} \in \mathbb{R}^{m \times n}$, with a probability $1 - 2N^{-p+1}$, we have*

$$\frac{mn}{|\Omega|} \|P_{T^\perp} R_\Omega P_T(\mathbf{H})\| \leq \|P_T(\mathbf{H})\|_\infty \sqrt{\frac{8\sigma pmn\mu a \log N}{3|\mathbf{\Omega}|}},$$

if $|\mathbf{\Omega}| \geq T_0$.

Proof. We write

$$P_{T^\perp} R_\Omega P_T(F) = \sum_{(i,j) \in \mathbf{\Omega}} \langle \mathbf{H}, P_T(\mathbf{e}_i \mathbf{e}_j^T) \rangle P_{T^\perp}(\mathbf{e}_i \mathbf{e}_j^T) = \sum_{(i,j) \in \mathbf{\Omega}} T_{i,j},$$

where $T_{i,j}(\mathbf{H}) = \langle \mathbf{H}, P_T(\mathbf{e}_i \mathbf{e}_j^T) \rangle P_{T^\perp}(\mathbf{e}_i \mathbf{e}_j^T)$. Evidently,

$$\mathbb{E}[P_{T^\perp} R_\Omega P_T(\mathbf{H})] = 0.$$

To use Lemma 3, we compute M and ρ^2 as,

$$
\begin{aligned}
M &= \max_{i \in [m] j \in [n]} \|T_{i,j}\| \\
&\leq \max_{i \in [m] j \in [n]} \max_{\|F\|_F = 1} \| \langle \mathbf{H}, P_T(\mathbf{e}_i \mathbf{e}_j^T) \rangle P_{T^\perp}(\mathbf{e}_i \mathbf{e}_j^T) \|_F \\
&\leq \max_{i \in [m] j \in [n]} \langle \mathbf{H}, P_T(\mathbf{e}_i \mathbf{e}_j^T) \rangle P_{T^\perp}(\mathbf{e}_i \mathbf{e}_j^T) \\
&\leq \|P_T(\mathbf{H})\|_\infty \max_{i \in [m] j \in [n]} \|P_{T^\perp}(\mathbf{e}_i \mathbf{e}_j^T)\|_F \\
&\leq \|P_T(\mathbf{H})\|_\infty \sqrt{\frac{\mu^2 \sigma^2(ab + r^2)}{mn}}
\end{aligned}
$$

and

$$
\begin{aligned}
\rho_{i,j}^2 &= \max\{\|\mathbb{E}[T_{i,j}, T_{i,j}^T]\|, \|\mathbb{E}[T_{i,j}^T, T_{i,j}]\|\} \\
&= \|P_T(\mathbf{H})\|_\infty^2 \max\{\|\mathbb{E}[P_{T^\perp}(\mathbf{e}_i \mathbf{e}_j^T)^T P_{T^\perp}(\mathbf{e}_i \mathbf{e}_j^T)]\|, \|\mathbb{E}[P_{T^\perp}(\mathbf{e}_i \mathbf{e}_j^T) P_{T^\perp}(\mathbf{e}_i \mathbf{e}_j^T)^T]\|\} \\
&= \|P_T(\mathbf{H})\|_\infty^2 \max\{\|\mathbb{E}[P_{Y^\perp} \mathbf{e}_j \mathbf{e}_i^T P_{X^\perp} \mathbf{e}_i \mathbf{e}_j^T P_{Y^\perp}]\|, \|\mathbb{E}[P_{X^\perp} \mathbf{e}_i \mathbf{e}_j^T P_{Y^\perp} \mathbf{e}_j \mathbf{e}_i^T P_{X^\perp}]\|\} \\
&\leq \|P_T(\mathbf{H})\|_\infty^2 \max\{\frac{\sigma\mu_{\mathbf{XY}} a}{m}\|\mathbb{E}[P_{Y^\perp} \mathbf{e}_j \mathbf{e}_j^T P_{Y^\perp}]\|, \frac{\sigma\mu_{\mathbf{XY}} b}{n}\|\mathbb{E}[P_{X^\perp} \mathbf{e}_i \mathbf{e}_i^T P_{X^\perp}]\|, \} \\
&\leq \|P_T(\mathbf{H})\|_\infty^2 \sigma \max\{\frac{\mu_{\mathbf{XY}} a}{m}\|P_{Y^\perp} \mathbb{E}[\mathbf{e}_j \mathbf{e}_j^T] P_{Y^\perp}\|, \frac{\mu_{\mathbf{XY}} b}{n}\|P_{X^\perp} \mathbb{E}[\mathbf{e}_i \mathbf{e}_i^T] P_{X^\perp}\|, \} \\
&\leq \|P_T(\mathbf{H})\|_\infty^2 \sigma \max\{\frac{\mu_{\mathbf{XY}} a}{mn}\|P_{Y^\perp} P_{Y^\perp}\|, \frac{\mu_{\mathbf{XY}} b}{mn}\|P_{X^\perp} P_{X^\perp}\|, \} \\
&\leq \|P_T(\mathbf{H})\|_\infty^2 \frac{\sigma\mu_{\mathbf{XY}} \max\{a, b\}}{mn}.
\end{aligned}
$$

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To prove simply without loss of the generality, we assume $b \leq a$, so we can get

$$\rho_{i,j}^2 \leq \|P_T(\mathbf{H})\|_\infty^2 \frac{\sigma\mu_{\mathbf{X}\mathbf{Y}}a}{mn} \leq \|P_T(\mathbf{H})\|_\infty^2 \frac{\sigma\mu a}{mn}$$

By Lemma 3, we have, if

$$\|P_T(\mathbf{H})\|_\infty^2 \frac{\sigma^2\mu^2(ab+r^2)}{mn} \log \frac{2N}{2N^{-p+1}} \leq \frac{3}{8}\|P_T(\mathbf{H})\|_\infty^2 \frac{\sigma\mu a|\mathbf{\Omega}|}{mn}$$

that is

$$\frac{8\sigma\mu(ab+r^2)p\log N}{3a} \leq |\mathbf{\Omega}|,$$

therefore, with a probability of $1 - 2N^{-p+1}$,

$$\begin{aligned}
\frac{mn}{|\mathbf{\Omega}|} &\leq \frac{mn}{|\mathbf{\Omega}|}\|P_T(\mathbf{H})\|_\infty \sqrt{\frac{8\sigma p\rho^2|\mathbf{\Omega}|\log N}{3}}\\
&\leq \frac{mn}{|\mathbf{\Omega}|}\|P_T(\mathbf{H})\|_\infty \sqrt{\frac{8\sigma p\mu a|\mathbf{\Omega}|\log N}{3mn}}\\
&= \|P_T(\mathbf{H})\|_\infty \sqrt{\frac{8\sigma mnp\mu a\log N}{3|\mathbf{\Omega}|}}
\end{aligned}$$

by Lemma 2, we need to use the condition $|\mathbf{\Omega}| \geq T_0$, and then

$$|\mathbf{\Omega}| \geq T_0 \geq \frac{32\sigma pr\mu^2(a+b)\log N}{3} \geq \frac{8\sigma p(ab+r^2)\mu\log N}{3a}$$

which is because $\mu \geq 1$, $a \geq b$, and $a \gg r$. Then under the condition $|\mathbf{\Omega}| \geq T_0$, we complete the proof. ∎

Lemma 8 *For a fixed $\mathbf{H} \in \mathbb{R}^{m \times n}$, with a probability $1 - 2N^{-p+2}$, we have*

$$\left\|\left(P_T - \frac{mn}{|\mathbf{\Omega}|P_T R_\mathbf{\Omega} P_T}\right)(\mathbf{H})\right\|_\infty \leq \sqrt{\frac{8\sigma pr\mu^2(a+b)\log N}{3|\mathbf{\Omega}|}}\|P_T(\mathbf{H})\|_\infty$$

and therefore if $|\mathbf{\Omega}| \leq T_0$,

$$\left\|\left(P_T - \frac{mn}{|\mathbf{\Omega}|P_T R_\mathbf{\Omega} P_T}\right)(\mathbf{H})\right\|_\infty \leq \frac{1}{2}\|P_T(\mathbf{F})\|_\infty$$

Proof. For each matrix index (a, b), sample (i, j) uniformly at random to define the random variable $\eta_{a,b} = [mnP_T R_\mathbf{\Omega} P_T(\mathbf{H}) - P_T(\mathbf{H})]$ We have

$$\mathbb{E}[\eta_{a,b}] = 0,$$

$$\eta_{a,b} \leq \|P_T R_{i,j} P_T - P_T\|\|P_T(\mathbf{H})\|_\infty \leq r\sigma\mu^2(a+b)\|P_T(\mathbf{H})\|_\infty$$

and

$$\begin{aligned}
\mathbb{E}[\eta_{a,b}^2] &= \mathbb{E}[([mnP_T R_{i,j} P_T(\mathbf{H}) - P_T(\mathbf{H})]_{a,b})^2]\\
&= \mathbb{E}[([m^2 n^2 P_T R_{i,j} P_T(\mathbf{H})]_{a,b})^2] + ([P_T(\mathbf{H})]_{a,b})^2 - 2mn\mathbb{E}[([P_T R_{i,j} P_T(\mathbf{H})]_{a,b}[P_T(\mathbf{H})]_{a,b})^2]\\
&= m^2 n^2 \mathbb{E}[([P_T R_{i,j} P_T(\mathbf{H})]_{a,b})^2] - ([P_T(\mathbf{H})]_{a,b})^2\\
&= m^2 n^2 \mathbb{E}[(\langle \mathbf{e}_a \mathbf{e}_b^T, P_T(\mathbf{e}_i \mathbf{e}_j^T)\rangle \langle \mathbf{H}, P_T(\mathbf{e}_i \mathbf{e}_j^T)\rangle)^2] - ([P_T(\mathbf{H})]_{a,b})^2\\
&= mn\|P_T(\mathbf{H})\|_F^2 \|P_T(\mathbf{e}_a \mathbf{e}_b)\|_F^2 - ([P_T(\mathbf{H})]_{a,b})^2\\
&\leq \|P_T(\mathbf{H})\|_\infty^2 r\sigma\mu^2(a+b)
\end{aligned}$$

Using the standard Bernstein Inequality, we have

$$\mathbb{P}\left[|[mnP_T R_\mathbf{\Omega} P_T(\mathbf{H}) - |\mathbf{\Omega}|P_T(\mathbf{H})]_{a,b}| > \sqrt{\frac{8|\mathbf{\Omega}|\|P_T(\mathbf{H})\|_\infty^2 r\sigma\mu^2(a+b)\log\frac{2}{2N^{-p}}}{3}}\right] \leq 2N^{-p}$$

Take the union bound, we have , with a probability of $1 - 2N^{-p+2}$

$$\|\frac{mn}{|\mathbf{\Omega}|}P_T R_\Omega P_T(\mathbf{H}) - P_T(\mathbf{H})\|_\infty \leq \sqrt{\frac{9\sigma r p \mu^2 (a+b) \ln N}{3|\mathbf{\Omega}|}} \|P_T(\mathbf{H})\|_\infty$$

If $|\mathbf{\Omega}| \geq T_0$, we have

$$\|\frac{mn}{|\mathbf{\Omega}|}P_T R_\Omega P_T(\mathbf{H})\|_\infty \leq \frac{1}{2}\|P_T(\mathbf{H})\|_\infty$$

∎

Next we need to verify that there exists a matrix $\mathbf{H}$ that satisfies the conditions in assumption A2, we follow the idea in [7] and construct F as follows. We generate a sequence of $\mathbf{H}_t, t = 1, ..., q$ as follows

$$\mathbf{H}_t = \frac{mn}{T_0}\sum_{i=1}^{t} R_{\Omega_i}(\mathbf{W}_i),$$

where $\mathbf{W}_1 = \mathbf{U}\mathbf{V}^T$ and $\mathbf{W}_{t+1}$ is defined as

$$\mathbf{W}_{t+1} = P_T(\mathbf{U}\mathbf{V}^T - \mathbf{H}_t) = (P_T - \frac{mn}{T_0}P_T R_{\Omega_t} P_T)(\mathbf{W}_t)$$

We randomly select qT_0 entries from Ω and partition the selected entries into q subsets as $\Omega_1, ..., \Omega_q$ with equal sizes, with $|\Omega_i| = T_0$, , $i = 1, ..., q$. Thus we have $\mathbf{H} = \mathbf{H}_q$ and $\mathbf{H} = R_\Omega(\mathbf{H})$.

Now we are ready to show that $\mathbf{H}$ satisfies the other two properties in assumption A2.

Lemma 9 *With a probability of $1 - 2qN^{-p+1}$, it is satisfied that*

$$\|P_T(\mathbf{H})\| \leq \sqrt{\frac{r}{2a}}$$

if $q \geq q_0$

Lemma 10 *With a probability of $1 - 2qN^{-p+1} - 2qN^{-p+2}$, it is satisfied that*

$$\|P_{T^\perp}(\mathbf{H})\| \leq \frac{1}{2}$$

if $q \geq q_0$

Proof. Because of Lemma 8 we have

$$\|\mathbf{H}_{t+1}\|_\infty = \|(P_T - \frac{mn}{T_0}P_T R_\Omega P_T)\mathbf{H}_t\|_\infty \leq \frac{1}{2}\|\mathbf{H}_t\|_\infty.$$

To bound $\|P_{T^\perp}(\mathbf{H})\|$, we have

$$\|P_{T^\perp}(\mathbf{H})\| \leq \sum_{i=1}^{q}\frac{mn}{T_0}\|P_{T^\perp}R_{\Omega_i}P_T(\mathbf{H}_i)\|$$

$$\leq \alpha \sum_{i=1}^{q}\|\mathbf{H}_i\|_\infty \leq \alpha\|\mathbf{H}_1\|_\infty \sum_{i=1}^{q}\frac{1}{2^{i-1}}$$

$$= 2\alpha\|\mathbf{H}_1\|_\infty \leq 2\sqrt{\frac{8\sigma p m n \mu a \log N}{3|\mathbf{\Omega}|}}\sqrt{\frac{\mu_1 r}{mn}}$$

$$\leq 2\sqrt{\frac{8\sigma p m n \mu a \log N}{3|\mathbf{\Omega}|}}$$

So when $|\mathbf{\Omega}| \geq \frac{128\sigma\mu_1 r p \mu a \log N}{3}$, it could be guaranteed that $\|P_{T^\perp}(\mathbf{H})\| \leq \frac{1}{2}$ when

$$|\mathbf{\Omega}| \geq \frac{128 p \sigma \mu \mu_{\mathbf{XY}} r (a+b) \log N}{3} = T_0.$$

∎

2 ϵ-Recovery Sampling Complexity

Consider the optimization problem below that if the perfect feature matrices $\mathbf{X}$ and $\mathbf{Y}$ are corrupted by $\Delta\mathbf{X}$ and $\Delta\mathbf{Y}$ and bounded by a constant $\|\Delta\mathbf{X}\|_F \leq s_1$ and $\|\Delta\mathbf{Y}\|_F \leq s_2$, so that we investigate the following relaxed optimization problem:

$$\min_{\mathbf{G}} \|R_\Omega((\mathbf{X}+\Delta\mathbf{X})^T\mathbf{G}(\mathbf{Y}+\Delta\mathbf{Y})-\mathbf{F})\|_F^2$$
$$\text{subject to } \mathbf{E}-\mathbf{X}^T\mathbf{G}\mathbf{Y} \in B(0,\phi), \tag{6}$$
$$\text{subject to } \|\mathbf{G}\|_1 \leq \alpha, \quad \|\mathbf{E}\|_* \leq \gamma.$$

where $B(0,\phi) \subset \mathbb{R}^{m\times n}$ is a ball with the radius of ϕ and center at 0.

The matrix $\mathbf{F}_{ij}$ is assumed to be observed partially i.i.d. from an index set $\{(i_\alpha, j_\alpha)\}_{\alpha=1}^m$ with unknown distribution.

We denote $\Theta = \{(\mathbf{G},\mathbf{E}) \mid \|\mathbf{G}\|_1 \leq \beta, \|\mathbf{E}\|_* \leq \gamma, \mathbf{E} = \mathbf{X}^T\mathbf{G}\mathbf{Y}\}$ as the feasible solution set, and $\theta = (\mathbf{G},\mathbf{E}) \in \Theta$ as any feasible solution. Let $F_\theta(i,j) = \mathbf{x}_i^T\mathbf{G}\mathbf{y}_j$ be the estimation function for $\mathbf{F}_{ij}$ with θ as the parameters, and $F_\Theta = \{f_\theta \mid \theta \in \Theta\}$ be the set of feasible functions. Denote the loss function as l where $l(f_\theta(i,j),\mathbf{F}_{ij}) = R_\Omega(\mathbf{X}^T\mathbf{G}\mathbf{Y}-\mathbf{F})_{i,j}^2$. Then, we introduce two "l-risk" quantities: the expected l-risk

$$\mathcal{R}_l(f) = \mathbb{E}_{(i,j)}[l(f_\theta(i,j),\mathbf{F}_{ij})],$$

and the empirical l-risk

$$\hat{\mathcal{R}}_l(f) = \frac{1}{s}\sum_{(i,j)}[l(f_\theta(i,j),\mathbf{F}_{ij})].$$

In this notation, our model is to solve for θ that parameterizes $f^* = \arg\min_{f\in F_\Theta} \hat{\mathcal{R}}_l(f)$, and it is sufficient to show that the recovery can be attained if $\hat{\mathcal{R}}_l(f^*)$ approaches to zero. Next we implement Rademacher complexity, a learning theoretic tool to measure the complexity of a function class. Then we will derive the sampling rate. To begin with, we cite the following Lemma [1] to bound the expected risk.

Lemma 11 (*Bound on Expected risk*). *Let l be a loss function with Lipschitz constant L_l in the compact domain respect to its first argument bounded by B, and p be a constant where $0 < p < 1$. Let $\mathfrak{R}(F_\Theta)$ be the Rademacher complexity of the function class F_Θ defined as:*

$$\mathfrak{R}(F_\Theta) = \mathbb{E}[\sup_{f\in F_\Theta} \frac{1}{s}\sum_{t=1}^s \omega_t l(f(i_t,j_t),\mathbf{F})] \tag{7}$$

where each ω_t takes values $\{\pm 1\}$ with equal probability. Then with the probability at least $1-p$, for all $f \in F_\Theta$ we have:

$$\mathcal{R}_l(f) \leq \hat{\mathcal{R}}_l(f) + 2\mathbb{E}[\mathfrak{R}(F_\Theta)] + B\sqrt{\frac{\log\frac{1}{p}}{2s}}. \tag{8}$$

In order to upper-bound $\mathcal{R}_l$, both $\hat{\mathcal{R}}_l$ and model complexity $\mathbb{E}_\Omega[\mathfrak{R}(F_\Theta)]$ need to be upper-bounded. The next key lemma shows that what affect the model complexity term $\mathbb{E}_\Omega[\mathfrak{R}(F_\Theta)]$ in matrix completion context.

The Rademacher complexity can be bounded in terms of β and γ by the following lemma:

Lemma 12 *Let $\mathcal{X} = \|\mathbf{X}\|_F$, $\mathcal{Y} = \|\mathbf{Y}\|_F$ and $d = \max(a,b)$,*

$$\mathbb{E}[\mathfrak{R}(F_\Theta)] \leq 2C_0 L_l \beta\mathcal{X}\mathcal{Y}\sqrt{\frac{\log 2d}{s}} + \sqrt{\frac{9dCL_l\alpha\sqrt{abp}(\sqrt{m}+\sqrt{n})}{s}}(s_1\mathcal{Y}+s_2\mathcal{X}+s_1s_2) \tag{9}$$

For proving clearly we firstly introduce Lemma 13 as below, which is a special case of Theorem 2 in [3];

Lemma 13 *Let* $S_\sigma = \{\mathbf{W} \in \mathbb{R}^{n \times n} \mid \|\mathbf{W}\|_* \leq \sigma\}$ *and* $a = \max_i \|\mathbf{A}_i\|_F$, *where* $\{\mathbf{A}_i \mid \mathbf{A}_i \in \mathbb{R}^{n \times n}\}_{i=1}^m$ *is an arbitrary set, then:*

$$\mathbb{E}[\sup_{W \in S_w} \frac{1}{m} \sum_{i=1}^m \omega_i \|\mathbf{W}\mathbf{A}_i\|_*] \leq 2a\sigma \sqrt{\frac{\log 2n}{m}}. \tag{10}$$

By using Lemma 13 and Rademacher contraction principle(e.g. Lemma in [4]), we can readily prove Lemma 12.

Proof. Denote $\mathbf{P} \in \mathbb{R}^{m \times n}$ with each entry $\mathbf{P}_{ij} = \sum_{\alpha: i_\alpha = i, j_\alpha = j} \omega_\alpha$, which means the 'hit-time' on the i, j-th element of Ω, then we can divide $\Re(F_\Theta)$ as:

$$\Re(F_\Theta) = \mathbb{E}_\sigma[\sup_{f \in F_\Theta} \frac{1}{s} \sum_{(i,j)} \mathbf{A}_{ij} l(f(i,j), \mathbf{F}_{ij})] + \mathbb{E}_\sigma[\sup_{f \in F_\Theta} \frac{1}{s} \sum_{(i,j)} \mathbf{B}_{ij} l(f(i,j), \mathbf{F}_{ij})] \tag{11}$$

In Eq. (11) we define

$$\mathbf{A}_{ij} = \begin{cases} \mathbf{P}_{ij}, & \text{if } h_{ij} > p \\ 0, & \text{otherwise.} \end{cases} \qquad \mathbf{B}_{ij} = \begin{cases} 0, & \text{if } h_{ij} > p \\ \mathbf{P}_{ij}, & \text{otherwise.} \end{cases}$$

where $h_{ij} = |\{\alpha : i_\alpha = i, j_\alpha = j\}|$ and p is a thresholding value discussed soon. Recall that $|l(f(i,j), \mathbf{F}_{ij})| \leq B$, from Lemma 10 in [6] we can infer that:

$$\mathbb{E}_\sigma[\sup_{f \in F_\Theta} \frac{1}{s} \sum_{(i,j)} \mathbf{A}_{ij} l(f(i,j), \mathbf{F}_{ij})] \leq \frac{B}{s} \mathbb{E}_\sigma[\sum_{(i,j)} |\mathbf{A}_{ij}|] \leq \frac{B}{\sqrt{p}} \tag{12}$$

Also we need to bound the other term in Eq. 11 below by using Lemma 13. We conduct that

$$\mathbb{E}_\sigma[\sup_{f \in F_\Theta} \frac{1}{s} \sum_{(i,j)} \mathbf{B}_{ij} l(f(i,j), \mathbf{F}_{ij})]$$

$$\leq \frac{L_l}{s} \mathbb{E}_\sigma[\sup_{\|\mathbf{G}\|_1 \leq \alpha} \sum_{(i,j)} \mathbf{B}_{ij} \mathbf{x}_i^T \mathbf{G} \mathbf{y}_j + \sup_{\|\mathbf{G}\|_1 \leq \alpha} \sum_{(i,j)} \mathbf{B}_{ij} \Delta \mathbf{x}_i^T \mathbf{G} \mathbf{y}_j + \tag{13}$$

$$\sup_{\|\mathbf{G}\|_1 \leq \alpha} \sum_{(i,j)} \mathbf{B}_{ij} \mathbf{x}_i^T \mathbf{G} \Delta \mathbf{y}_j + \sup_{\|\mathbf{G}\|_1 \leq \alpha} \sum_{(i,j)} \mathbf{B}_{ij} \Delta \mathbf{x}_i^T \mathbf{G} \Delta \mathbf{y}_j]$$

Since $\|\mathbf{G}\|_* \leq C_0 \|\mathbf{G}\|_2 \leq C_0 \|\mathbf{G}\|_1$ as the matrix-norm equivalence for any $\mathbf{G} \in \mathbb{R}^{a \times b}$ while there always exists a fixed C_0, for the last three terms we can use Holder's inequality to upper-bound it as below:

$$\frac{L_l}{s}[\sup_{\|\mathbf{G}\|_1 \leq \alpha} \sum_{(i,j)} \mathbf{B}_{ij} \Delta \mathbf{x}_i^T \mathbf{G} \mathbf{y}_j + \sup_{\|\mathbf{G}\|_1 \leq \alpha} \sum_{(i,j)} \mathbf{B}_{ij} \mathbf{x}_i^T \mathbf{G} \Delta \mathbf{y}_j + \sup_{\|\mathbf{G}\|_1 \leq \alpha} \sum_{(i,j)} \mathbf{B}_{ij} \Delta \mathbf{x}_i^T \mathbf{G} \Delta \mathbf{y}_j]$$

$$\leq \frac{L_l \mathbb{E}[\|\mathbf{B}\|_2]}{s} \sup_{\|\mathbf{G}\|_1 \leq \alpha} [\|\Delta \mathbf{X}^T \mathbf{G} \mathbf{Y}\|_* + \sup_{\|\mathbf{G}\|_1 \leq \alpha} \|\mathbf{X}^T \mathbf{G} \Delta \mathbf{Y}\|_* + \sup_{\|\mathbf{G}\|_1 \leq \alpha} \|\Delta \mathbf{X}^T \mathbf{G} \Delta \mathbf{Y}\|_*]$$

$$\leq \frac{\sqrt{ab}\alpha L \mathbb{E}[\|\mathbf{B}\|_2]}{s} [\|\Delta \mathbf{X}^T\|_F \|\mathbf{Y}\|_F + \|\mathbf{X}^T\|_F \|\Delta \mathbf{Y}\|_F + \|\Delta \mathbf{X}^T\|_F \|\Delta \mathbf{Y}\|_F] \tag{14}$$

$$\leq \frac{\sqrt{ab}\alpha L}{s} (s_1 \mathcal{Y} + s_2 \mathcal{X} + s_1 s_2) \mathbb{E}[\|\mathbf{B}\|_2]$$

$$\leq \frac{2.2CL\alpha\sqrt{abp}(\sqrt{m} + \sqrt{n})}{s} (s_1 \mathcal{Y} + s_2 \mathcal{X} + s_1 s_2)$$

where the last inequality is from Lemma 1 in [6].

Next we bound the term $\mathbb{E}_\sigma[\sup_{\|\mathbf{G}\|_1 \leq \alpha} \sum_{(i,j)} \mathbf{B}_{ij} \mathbf{x}_{i_\alpha}^T \mathbf{G} \mathbf{y}_{j_\alpha}]$ in Eq. (13) as:

$$\frac{L_l}{s} \mathbb{E}_\sigma[\sup_{\|\mathbf{G}\|_1 \leq \alpha} \sum_{\alpha=1}^s \omega_\alpha \mathbf{x}_{i_\alpha}^T \mathbf{G} \mathbf{y}_{j_\alpha}] \leq L_l \mathbb{E}[\sup_{\|\mathbf{G}\|_1 \leq \alpha} \frac{1}{s} \sum_{\alpha=1}^s \omega_\alpha tr(\mathbf{x}_{i_\alpha}^T \mathbf{G} \mathbf{y}_{j_\alpha})]$$

$$\leq L_l \mathbb{E}[\sup_{\|\mathbf{G}\|_1 \leq \alpha} \frac{1}{s} \sum_{\alpha=1}^s \omega_\alpha tr(\mathbf{G} \mathbf{y}_{j_\alpha} \mathbf{x}_{i_\alpha}^T)] \leq 2C_0 L_l \alpha \max_{i,j} \|\mathbf{y}_j \mathbf{x}_i^T\|_2 \sqrt{\frac{\log 2d}{s}} \qquad (15)$$

$$\leq 2C_0 L_l \alpha \mathcal{X} \mathcal{Y} \sqrt{\frac{\log 2d}{s}}$$

Combining the above bounds in Eq. (12), Eq. (14) and Eq. (15) together, with $p = (sB)/[2.2CL\alpha\sqrt{abp}(\sqrt{m} + \sqrt{n})(s_1\mathcal{Y} + s_2\mathcal{X} + s_1 s_2)]$ we can get the bound for $\mathbb{E}[\mathfrak{R}(F_\Theta)]$ as:

$$\mathbb{E}[\mathfrak{R}(F_\Theta)] \leq 2C_0 L_l \alpha \mathcal{X} \mathcal{Y} \sqrt{\frac{\log 2d}{s}} + \frac{9CL_l \alpha \sqrt{abp}(\sqrt{m} + \sqrt{n})}{s}(s_1\mathcal{Y} + s_2\mathcal{X} + s_1 s_2) \qquad (16)$$

∎

Lemma 15 clarifies the upper-bound of the complexity of f. Additionally, with proper chosen λ_E and λ_G, the empirical risk $\hat{\mathcal{R}}(f)$ can be sufficiently small. Therefore we conclude the upper bound of $\mathcal{R}(f^*)$ as below.

Lemma 14 *With a probability at least $1 - p$, thje expected l-risk of an optimal solution will be bounded by:*

$$\mathcal{R}(f^*) \leq 2C_0 L_l \alpha \mathcal{X} \mathcal{Y} \sqrt{\frac{\log 2d}{s}} + \frac{18CL_l \alpha \sqrt{abpN}}{s}(s_1\mathcal{Y} + s_2\mathcal{X} + s_1 s_2) + B\sqrt{\frac{\log \frac{1}{p}}{2s}}$$

Now consider another view to upper-bound our model, then we give Lemma 12 as followed,

Lemma 15 *Let $\mathcal{X} = \|\mathbf{X}\|_F$, $\mathcal{Y} = \|\mathbf{Y}\|_F$ and $d = \max(a, b)$,*

$$\mathbb{E}[\mathfrak{R}(F_\Theta)] \leq 2C_0 L_l [\gamma \sqrt{\frac{\log 2N}{s}} + \phi \sqrt{\frac{\log 2N}{s}} + \alpha \sqrt{\frac{\log 2d}{s}}(s_1\mathcal{Y} + s_2\mathcal{X} + s_1 s_2)] \qquad (17)$$

Again, by using Lemma 13 and Rademacher contraction principle(e.g. Lemma in [4]), we can prove Lemma 15.

Proof. $\mathbb{E}(\mathfrak{R}(F_\Theta))$ can be bounded as above, we have

$$\mathbb{E}_\sigma[\sup_{f \in F_\Theta} \frac{1}{s} \sum_{\alpha=1}^s \omega_\alpha l(f(i_\alpha, j_\alpha), \mathbf{F}_{i_\alpha j_\alpha})]$$

$$\leq \frac{L_l}{s} \mathbb{E}_\sigma[\sup_{\|\mathbf{G}\|_1 \leq \alpha} \sum_{\alpha=1}^s \omega_\alpha \mathbf{x}_{i_\alpha}^T \mathbf{G} \mathbf{y}_{j_\alpha} + \sup_{\|\mathbf{G}\|_1 \leq \alpha} \sum_{\alpha=1}^s \omega_\alpha \Delta\mathbf{x}_{i_\alpha}^T \mathbf{G} \mathbf{y}_{j_\alpha} + \qquad (18)$$

$$\sup_{\|\mathbf{G}\|_1 \leq \alpha} \sum_{\alpha=1}^s \omega_\alpha \mathbf{x}_{i_\alpha}^T \mathbf{G} \Delta\mathbf{y}_{j_\alpha} + \sup_{\|\mathbf{G}\|_1 \leq \alpha} \sum_{\alpha=1}^s \omega_\alpha \Delta\mathbf{x}_{i_\alpha}^T \mathbf{G} \Delta\mathbf{y}_{j_\alpha}]$$

Then one can follow the same approach in Eq. (15) as

$$\frac{L_l}{s}\mathbb{E}_\sigma[\sup_{\|\mathbf{G}\|_1\leq\alpha}\sum_{\alpha=1}^s\omega_\alpha\mathbf{x}_i^T\mathbf{G}\mathbf{y}_j+\sup_{\|\mathbf{G}\|_1\leq\alpha}\sum_{\alpha=1}^s\omega_\alpha\Delta\mathbf{x}_i^T\mathbf{G}\mathbf{y}_j+$$

$$\sup_{\|\mathbf{G}\|_1\leq\alpha}\sum_{\alpha=1}^s\omega_\alpha\mathbf{x}_i^T\mathbf{G}\Delta\mathbf{y}_j+\sup_{\|\mathbf{G}\|_1\leq\alpha}\sum_{\alpha=1}^s\omega_\alpha\Delta\mathbf{x}_i^T\mathbf{G}\Delta\mathbf{y}_j]$$

$$\leq L_l\mathbb{E}[\frac{1}{s}(\sup_{\|\mathbf{E}\|_*\leq\gamma}\sum_{\alpha=1}^s\omega_\alpha tr(\mathbf{E}_{i_\alpha j_\alpha}\mathbf{e}_{j_\alpha}\mathbf{e}_{i_\alpha}^T)+\sup_{\|\mathbf{\Phi}\|_F\leq\phi}\sum_{\alpha=1}^s\omega_\alpha tr(\mathbf{\Phi}_{i_\alpha j_\alpha}\mathbf{e}_{j_\alpha}\mathbf{e}_{i_\alpha}^T))+ \qquad (19)$$

$$2L_l\alpha r\sqrt{\frac{d\log 2d}{s}}[\max_{i,j}\|\mathbf{y}_j\Delta\mathbf{x}_i^T\|_2+\max_{i,j}\|\Delta\mathbf{y}_j\mathbf{x}_i^T\|_2+\max_{i,j}\|\Delta\mathbf{y}_j\Delta\mathbf{x}_i^T\|_2]$$

$$\leq 2L_l[\gamma\sqrt{\frac{\log 2N}{s}}+C_0\phi\sqrt{\frac{\log 2N}{s}}+C_0\alpha\sqrt{\frac{\log 2d}{s}}(s_1\mathcal{Y}+s_2\mathcal{X}+s_1 s_2)]$$

$$\leq 2C_0 L_l[\gamma\sqrt{\frac{\log 2N}{s}}+\phi\sqrt{\frac{\log 2N}{s}}+\alpha\sqrt{\frac{\log 2d}{s}}(s_1\mathcal{Y}+s_2\mathcal{X}+s_1 s_2)]$$

where the last equation is derived by applying Lemma 13. So we derive another upper bound of $\mathbb{E}[\mathfrak{R}(F_\Theta)]$ as

$$\mathbb{E}[\mathfrak{R}(F_\Theta)]\leq 2C_0 L_l[\gamma\sqrt{\frac{\log 2N}{s}}+\phi\sqrt{\frac{\log 2N}{s}}+\alpha\sqrt{\frac{\log 2d}{s}}(s_1\mathcal{Y}+s_2\mathcal{X}+s_1 s_2)] \qquad (20)$$

∎

Then our Theorem 2 can be attained directly from Lemma 14 and Lemma 15.

Theorem 2 *Denote $\|\mathbf{E}\|_*\leq\alpha$, $\|\mathbf{G}\|_1\leq\gamma$, and the perfect side feature matrices (containing latent features of $\mathbf{F}$) are corrupted with $\Delta\mathbf{X}$ and $\Delta\mathbf{Y}$ where $\|\Delta\mathbf{X}\|_F\leq s_1, \|\Delta\mathbf{Y}\|_F\leq s_2$ and $S=\max(s_1,s_2)$. To ϵ-recover $\mathbf{F}$ that the expected loss $\mathbb{E}[l(f,\mathbf{F})]<\epsilon$ for a given arbitrarily small $\epsilon>0$, $O(\min((\gamma^2+\phi^2)\log N, S^2\alpha\sqrt{N})/\epsilon^2)$ observations are sufficient for our model to achieve an ϵ-recovery when corrupted factors of side information are bounded.*

For the goal of investigating the recovery guarantee under the generalized frame of our work, it is noted that we can replace any norm-regularizers $\|\mathbf{G}\|_\sim$ of $\mathbf{G}$ satisfying that $\|\mathbf{G}\|_\sim\leq\|\mathbf{G}\|_1$. Therefore it is feasible to further explore more structural priors in various situation.

3 Convergence Analysis

In this subsection, we present the proof of the global convergence for our algorithm.

For conveniently writing, we write the Lagrangian function of our problem as

$$\mathcal{L}(\mathbf{E},\mathbf{G},\mathbf{C},\mathbf{M}_1,\mathbf{M}_2,\beta)=\frac{1}{2}\|\mathbf{C}\|_F^2+\lambda_\mathbf{E}\|\mathbf{E}\|_*+\lambda_\mathbf{G}\|\mathbf{G}\|_1+$$

$$\langle\mathbf{M},\mathcal{B}(\mathbf{E})+\mathcal{A}(\mathbf{G})+\mathcal{N}(\mathbf{C})-\mathbf{D}\rangle+\frac{\beta}{2}\|\mathcal{B}(\mathbf{E})+\mathcal{A}(\mathbf{G})+\mathcal{N}(\mathbf{C})-\mathbf{D}\|_F^2 \qquad (21)$$

where $\mathcal{B}(\mathbf{E})=\begin{pmatrix}\Omega(\mathbf{E})\\\mathbf{E}\end{pmatrix}$, $\mathcal{A}(\mathbf{G})=\begin{pmatrix}0\\-\mathbf{X}^T\mathbf{G}\mathbf{Y}\end{pmatrix}$, $\mathcal{N}(\mathbf{C})=\begin{pmatrix}0\\\mathbf{C}\end{pmatrix}$ and $\mathbf{D}=\begin{pmatrix}\Omega(\mathbf{F})\\0\end{pmatrix}$. $\mathbf{M}$ is the multiplier stacked as $\begin{pmatrix}\mathbf{M}_1\\\mathbf{M}_2\end{pmatrix}$.

The proving framework consists of three steps: The first step includes Lemma 16 for the proof of Lemma 17 and Theorem 3; the next step is the proof of Lemma 17 which indicates the convergence of our algorithm; the third step is to clarify our algorithm converges to a KKT point of problem (4), which is also the global minimizer for convex problem, shown in Theorem 3.

Lemma 16 *Let $\mathbf{G}^k$, $\mathbf{E}^k$, $\mathbf{C}^k$ be the optimal solution for each individual subproblem at the k-th iteration, then it satisfies that $-\beta_k\tau_A(\mathbf{G}^{k+1}-\mathbf{G}^k)-\mathcal{A}^*(\bar{\mathbf{M}}^{k+1})\in\partial\|\mathbf{G}^{k+1}\|_1, -\beta_k\tau_B(\mathbf{E}^{k+1}-$*

$\mathbf{E}^k) - \mathcal{B}^*(\hat{\mathbf{M}}^{k+1}) \in \partial\|\mathbf{E}^{k+1}\|_*$ *where* $\bar{\mathbf{M}}^{k+1} = \mathbf{M}^k + \beta_k[\mathcal{A}(\mathbf{G}^k) + \mathcal{B}(\mathbf{E}^{k+1}) + \mathcal{N}(\mathbf{C}^k) - \mathbf{D}]$, $\hat{\mathbf{M}}^{k+1} = \mathbf{M}^k + \beta_k[\mathcal{A}(\mathbf{G}^{k+1}) + \mathcal{B}(\mathbf{E}^{k+1}) + \mathcal{N}(\mathbf{C}^k) - \mathbf{D}]$, *here* $\partial\|\cdot\|$ *denotes the subgradient of an arbitrary* $\|\cdot\|$, *and* $\mathcal{A}^*$ *is the adjoint operator of* $\mathcal{A}$.

Note that $\mathcal{A}^* = \mathbf{A}^T$ if $\mathcal{A}$ is a linear operator while $\mathcal{A}(\mathbf{X}) = \mathbf{A}\mathbf{X}$. This Lemma is directly derived from the optimality conditions of subproblems when solving $\mathbf{G}$ and $\mathbf{E}$ individually.

Next we present the lemma implying the convergence.

Lemma 17 *Given* β_k *is non-decreasing and upper bounded,* $\tau_A > \|\mathcal{A}\|^2$, $\tau_B > \|\mathcal{B}\|^2$, *and* $(\mathbf{G}^*, \mathbf{E}^*, \mathbf{C}^*, \mathbf{M}^*)$ *is any KKT point of problem 21, then:*

$\{\tau_A\|\mathbf{G}^k - \mathbf{G}^*\|_F^2 - \|\mathcal{A}(\mathbf{G}^k - \mathbf{G}^*)\|_F^2 + \tau_B\|\mathbf{E}^k - \mathbf{E}^*\|_F^2 + \|\mathbf{C}^k - \mathbf{C}^*\|_F - \|\mathcal{N}(\mathbf{C}^k - \mathbf{C}^*)\|_F^2 +$
$\beta_k^{-2}\|\mathbf{M}^k - \mathbf{M}^*\|_F^2\}$*is non-increasing; and*

$$\|\mathbf{G}^k - \mathbf{G}^{k+1}\|_F^2 \to 0, \|\mathbf{E}^k - \mathbf{E}^{k+1}\|_F^2 \to 0, \|\mathbf{C}^k - \mathbf{C}^{k+1}\|_F^2 \to 0, \|\mathbf{M}^k - \mathbf{M}^{k+1}\|_F^2 \to 0.$$
$$(22)$$

For proving the non-increase property of the first sequence, it is equivalent to investigate the following inequality:

$$\begin{aligned}
&\tau_A\|\mathbf{G}^{k+1} - \mathbf{G}^*\|_F^2 - \|\mathcal{A}(\mathbf{G}^{k+1} - \mathbf{G}^*)\|_F^2 + \tau_B\|\mathbf{E}^{k+1} - \mathbf{E}^*\|_F^2 + \|\mathbf{C}^{k+1} - \mathbf{C}^*\|_F \\
&- \|\mathcal{N}(\mathbf{C}^{k+1} - \mathbf{C}^*)\|_F^2 + \beta_k^{-2}\|\mathbf{M}^{k+1} - \mathbf{M}^*\|_F^2 - (\tau_A\|\mathbf{G}^k - \mathbf{G}^*\|_F^2 - \|\mathcal{A}(\mathbf{G}^k - \mathbf{G}^*)\|_F^2 \quad (23) \\
&+ \tau_B\|\mathbf{E}^k - \mathbf{E}^*\|_F^2 + \|\mathbf{C}^k - \mathbf{C}^*\|_F - \|\mathcal{N}(\mathbf{C}^k - \mathbf{C}^*)\|_F^2 + \beta_k^{-2}\|\mathbf{M}^k - \mathbf{M}^*\|_F^2) \le 0
\end{aligned}$$

For proving the above inequality, we list several facts to be used:

$$\begin{aligned}
&\mathbf{M}^{k+1} = \mathbf{M}^k + \beta_k(\mathcal{A}(\mathbf{G}^{k+1}) + \mathcal{B}(\mathbf{E}^{k+1}) + \mathcal{N}(\mathbf{C}^{k+1}) - \mathbf{D}), \\
&2\left\langle \mathbf{G}^{k+1} - \mathbf{G}^*, \mathbf{G}^{k+1} - \mathbf{G}^k \right\rangle = \|\mathbf{G}^{k+1} - \mathbf{G}^*\|_F^2 - \|\mathbf{G}^k - \mathbf{G}^*\|_F^2 + \|\mathbf{G}^{k+1} - \mathbf{G}^k\|_F^2, \quad (24) \\
&\mathcal{A}(\mathbf{G}^*) + \mathcal{B}(\mathbf{E}^*) + \mathcal{N}(\mathbf{C}^*) - \mathbf{D} = 0, \\
&\langle \mathbf{M}, \mathcal{A}(\mathbf{G}) \rangle = \langle \mathcal{A}^*(\mathbf{M}), \mathbf{G} \rangle, \langle \mathbf{M}, \mathcal{B}(\mathbf{E}) \rangle = \langle \mathcal{B}^*(\mathbf{M}), \mathbf{E} \rangle.
\end{aligned}$$

Proof.

$$\begin{aligned}
&\tau_A\|\mathbf{G}^{k+1} - \mathbf{G}^*\|_F^2 - \|\mathcal{A}(\mathbf{G}^{k+1} - \mathbf{G}^*)\|_F^2 + \tau_B\|\mathbf{E}^{k+1} - \mathbf{E}^*\|_F^2 + \|\mathbf{C}^{k+1} - \mathbf{C}^*\|_F - \|\mathcal{N}(\mathbf{C}^{k+1} - \mathbf{C}^*)\|_F^2 + \\
&\beta_k^{-2}\|\mathbf{M}^{k+1} - \mathbf{M}^*\|_F^2 - (\tau_A\|\mathbf{G}^k - \mathbf{G}^*\|_F^2 - \|\mathcal{A}(\mathbf{G}^k - \mathbf{G}^*)\|_F^2 + \tau_B\|\mathbf{E}^k - \mathbf{E}^*\|_F^2 + \|\mathbf{C}^k - \mathbf{C}^*\|_F \\
&- \|\mathcal{N}(\mathbf{C}^k - \mathbf{C}^*)\|_F^2 + \beta_k^{-2}\|\mathbf{M}^k - \mathbf{M}^*\|_F^2) \\
=&2\tau_A\left\langle \mathbf{G}^{k+1} - \mathbf{G}^*, \mathbf{G}^{k+1} - \mathbf{G}^k \right\rangle - \tau_A\|\mathbf{G}^{k+1} - \mathbf{G}^k\|_F^2 - 2\left\langle \mathcal{A}(\mathbf{G}^{k+1} - \mathbf{G}^*), \mathcal{A}(\mathbf{G}^{k+1} - \mathbf{G}^k) \right\rangle + \\
&\|\mathcal{A}(\mathbf{G}^{k+1} - \mathbf{G}^k)\|_F^2 + 2\tau_B\left\langle \mathbf{E}^{k+1} - \mathbf{E}^*, \mathbf{E}^{k+1} - \mathbf{E}^k \right\rangle - \tau_B\|\mathbf{E}^{k+1} - \mathbf{E}^k\|_F^2 + \\
&2\tau_N\left\langle \mathbf{C}^{k+1} - \mathbf{C}^*, \mathbf{C}^{k+1} - \mathbf{C}^k \right\rangle - \tau_A\|\mathbf{C}^{k+1} - \mathbf{C}^k\|_F^2 - 2\left\langle \mathcal{N}(\mathbf{C}^{k+1} - \mathbf{C}^*), \mathcal{N}(\mathbf{C}^{k+1} - \mathbf{C}^k) \right\rangle + \\
&\|\mathcal{N}(\mathbf{C}^{k+1} - \mathbf{C}^k)\|_F^2 \\
=&- \{\beta_k^{-2}\|\mathbf{M}^{k+1} - \mathbf{M}^k\|_F^2 + \tau_B\|\mathbf{E}^{k+1} - \mathbf{E}^k\|_F - 2\beta_k^{-1}\left\langle \mathbf{M}^{k+1} - \mathbf{M}^k, \mathcal{B}(\mathbf{E}^{k+1} - \mathbf{E}^k) \right\rangle\} - \\
&(\tau_A\|\mathbf{G}^{k+1} - \mathbf{G}^k\|_F^2 - \|\mathcal{A}(\mathbf{G}^{k+1} - \mathbf{G}^k)\|_F^2) - (\|\mathbf{C}^{k+1} - \mathbf{C}^k\|_F^2 - \|\mathcal{N}(\mathbf{C}^{k+1} - \mathbf{C}^k)\|_F^2) - \\
&2\beta_k^{-1}\left\langle \mathbf{G}^{k+1} - \mathbf{G}^*, [-\beta_k\tau_A(\mathbf{G}^{k+1} - \mathbf{G}^k) - \mathcal{A}^*(\bar{\mathbf{M}}^{k+1})] + \mathcal{A}^*(\mathbf{M}^*) \right\rangle - \\
&2\beta_k^{-1}\left\langle \mathbf{E}^{k+1} - \mathbf{E}^*, [-\beta_k\tau_B(\mathbf{E}^{k+1} - \mathbf{E}^k) - \mathcal{B}^*(\hat{\mathbf{M}}^{k+1})] + \mathcal{B}^*(\mathbf{M}^*) \right\rangle - \\
&2\beta_k^{-1}\left\langle \mathbf{C}^{k+1} - \mathbf{C}^*, [-\beta_k(\mathbf{C}^{k+1} - \mathbf{C}^k) - \mathcal{N}^*(\mathbf{M}^{k+1})] + \mathcal{N}^*(\mathbf{M}^*) \right\rangle
\end{aligned}$$
$$(25)$$

Since $\tau_A \ge \|\mathcal{A}\|^2$, we can check that

$$\tau_A\|\cdot\|_F^2 - \|\mathcal{A}(\cdot)\|_F^2 \ge 0.$$

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and similarly it is clear that

$$\beta_k^{-2}\|\mathbf{M}^{k+1}-\mathbf{M}^k\|_F^2+\tau_B\|\mathbf{E}^{k+1}-\mathbf{E}^k\|_F^2-2\beta_k^{-1}\left\langle\mathbf{M}^{k+1}-\mathbf{M}^k,\mathcal{B}(\mathbf{E}^{k+1}-\mathbf{E}^k)\right\rangle\geq 0$$

The last three terms in Eq. (25) are nonnegative due to Lemma 16 and the monotonicity of subgradient mapping. So the non-increasing property in Lemma 17 is proved. Because of the non-increasing property and non-negativity, it has a limit. Then we can see that

$$\tau_A\|\mathbf{G}^{k+1}-\mathbf{G}^k\|_F^2-\|\mathcal{A}(\mathbf{G}^{k+1}-\mathbf{G}^k)\|_F^2\to 0,$$

$$\|\mathbf{C}^{k+1}-\mathbf{C}^k\|_F^2-\|\mathcal{N}(\mathbf{C}^{k+1}-\mathbf{C}^k)\|_F^2\to 0.$$

$$\beta_k^{-2}\|\mathbf{M}^{k+1}-\mathbf{M}^k\|_F^2+\tau_B\|\mathbf{E}^{k+1}-\mathbf{E}^k\|_F^2-2\beta_k^{-1}\left\langle\mathbf{M}^{k+1}-\mathbf{M}^k,\mathcal{B}(\mathbf{E}^{k+1}-\mathbf{E}^k)\right\rangle\to 0$$

due to their non-negativity. So $\|\mathbf{G}^{k+1}-\mathbf{G}^k\|_F\to 0$ and $\|\mathbf{C}^{k+1}-\mathbf{C}^k\|_F\to 0$ can be obtained from the first two limits. Note that

$$\beta_k^{-2}\|\mathbf{M}^{k+1}-\mathbf{M}^k\|_F^2+\tau_B\|\mathbf{E}^{k+1}-\mathbf{E}^k\|_F^2-2\beta_k^{-1}\left\langle\mathbf{M}^{k+1}-\mathbf{M}^k,\mathcal{B}(\mathbf{E}^{k+1}-\mathbf{E}^k)\right\rangle$$
$$\geq\beta_k^{-2}\|\mathbf{M}^{k+1}-\mathbf{M}^k\|_F^2+\tau_B\|\mathbf{E}^{k+1}-\mathbf{E}^k\|_F^2-2\beta_k^{-1}\|\mathbf{M}^{k+1}-\mathbf{M}^k\|_F\|\mathcal{B}(\mathbf{E}^{k+1}-\mathbf{E}^k)\|_F$$
$$=(\beta_k^{-1}\|\mathbf{M}^{k+1}-\mathbf{M}^k\|_F-\|\mathcal{B}(\mathbf{E}^{k+1}-\mathbf{E}^k)\|_F)^2+\tau_B\|\mathbf{E}^{k+1}-\mathbf{E}^k\|_F^2-\|\mathcal{B}(\mathbf{E}^{k+1}-\mathbf{E}^k)\|_F^2$$
$$\geq\tau_B\|\mathbf{E}^{k+1}-\mathbf{E}^k\|_F^2-\|\mathcal{B}(\mathbf{E}^{k+1}-\mathbf{E}^k)\|_F^2\geq 0.$$

$$(26)$$

So we have that $\|\mathbf{E}^{k+1}-\mathbf{E}^k\|_F\to 0$. Furthermore,

$$\beta_k^{-2}\|\mathbf{M}^{k+1}-\mathbf{M}^k\|_F^2+\tau_B\|\mathbf{E}^{k+1}-\mathbf{E}^k\|_F^2-2\beta_k^{-1}\left\langle\mathbf{M}^{k+1}-\mathbf{M}^k,\mathcal{B}(\mathbf{E}^{k+1}-\mathbf{E}^k)\right\rangle$$
$$(\beta_k^{-1}\|\mathbf{M}^{k+1}-\mathbf{M}^k\|_F-\sqrt{\tau_B}\|\mathbf{E}^{k+1}-\mathbf{E}^k\|_F)^2+$$
$$2\beta_k^{-1}(\sqrt{\tau_B}\|\mathbf{M}^{k+1}-\mathbf{M}^k\|_F\|\mathbf{E}^{k+1}-\mathbf{E}^k\|_F-\left\langle\mathbf{M}^{k+1}-\mathbf{M}^k,\mathcal{B}(\mathbf{E}^{k+1}-\mathbf{E}^k)\right\rangle)$$
$$\geq(\beta_k^{-1}\|\mathbf{M}^{k+1}-\mathbf{M}^k\|_F-\sqrt{\tau_B}\|\mathbf{E}^{k+1}-\mathbf{E}^k\|_F)^2.$$

$$(27)$$

So $\beta_k^{-2}\|\mathbf{M}^{k+1}-\mathbf{M}^k\|_F^2+\tau_B\|\mathbf{E}^{k+1}-\mathbf{E}^k\|_F^2-2\beta_k^{-1}\left\langle\mathbf{M}^{k+1}-\mathbf{M}^k,\mathcal{B}(\mathbf{E}^{k+1}-\mathbf{E}^k)\right\rangle\to 0$. This results in $\|\mathbf{M}^{k+1}-\mathbf{M}^k\|_F\to 0$ noting that $\|\mathbf{E}^{k+1}-\mathbf{E}^k\|_F\to 0$. ∎

Based on Lemma 16 and Lemma 17, we can derive the following theorem.

Theorem 3 *If β_k is non-decreasing and upper-bounded, $\tau_A>\|\mathcal{A}\|$, and $\tau_B>\|\mathcal{B}\|$ then the sequence $\{(\mathbf{C}^k,\mathbf{G}^k,\mathbf{E}^k,\mathbf{M}^k)\}$ generated by adaptive LADMM converges to a KKT point of problem (4).*

Proof. By Lemma 17, $\{(\mathbf{C}^k,\mathbf{G}^k,\mathbf{E}^k,\mathbf{M}^k)\}$ is bounded, hence there is a subsequence that $(\mathbf{C}^{k_i},\mathbf{G}^{k_i},\mathbf{E}^{k_i},\mathbf{M}^{k_i})\to(\mathbf{C}^\infty,\mathbf{G}^\infty,\mathbf{E}^\infty,\mathbf{M}^\infty)$. We accomplish the proof in two steps.

We first prove that $(\mathbf{C}^\infty,\mathbf{G}^\infty,\mathbf{E}^\infty,\mathbf{M}^\infty)$ is a KKT point of our optimization problem.

By Lemma 17, $\mathcal{A}(\mathbf{G}^{k+1})+\mathcal{B}(\mathbf{E}^{k+1})+\mathcal{N}(\mathbf{C}^{k+1})-\mathbf{D}=\beta_k^{-1}(\mathbf{M}^{k+1}-\mathbf{M}^k)\to 0$. This shows that any accumulation point of $\{(\mathbf{C}^k,\mathbf{G}^k,\mathbf{E}^k,\mathbf{M}^k)\}$ is a feasible solution.

Without the loss of generality, suppose $\lambda_G=\lambda_E=\frac{1}{2}$. by letting $k=k_i-1$ in Lemma 16 and the subgradient definition, we have

$$\|\mathbf{G}^{k_i}\|_1+\|\mathbf{E}^{k_i}\|_*+\|\mathbf{C}^{k_i}\|_F$$
$$\leq\|\mathbf{G}^*\|_1+\|\mathbf{E}^*\|_*+\|\mathbf{C}^*\|_F+\left\langle\mathbf{G}^{k_i}-\mathbf{G}^*,-\beta_{k_i-1}\tau_A(\mathbf{G}^{k_i}-\mathbf{G}^{k_i-1})-\mathcal{A}^*(\bar{\mathbf{M}}^{k_i})\right\rangle$$
$$+\left\langle\mathbf{E}^{k_i}-\mathbf{E}^*,-\beta_{k_i-1}\tau_B(\mathbf{E}^{k_i}-\mathbf{E}^{k_i-1})-\mathcal{B}^*(\hat{\mathbf{M}}^{k_i})\right\rangle+\left\langle\mathbf{C}^{k_i}-\mathbf{C}^*,-\beta_{k_i-1}(\mathbf{C}^{k_i}-\mathbf{C}^{k_i-1})-\mathcal{N}^*(\mathbf{M}^{k_i})\right\rangle$$

$$(28)$$

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Suppose $i \to \infty$, from Lemma 17, we can observe $\mathbf{G}^{k_i} - \mathbf{G}^{k_i-1} \to 0$ so that

$$\|\mathbf{G}^\infty\|_1 + \|\mathbf{E}^\infty\|_* + \|\mathbf{C}^\infty\|_F^2$$
$$\leq \|\mathbf{G}^*\|_1 + \|\mathbf{E}^*\|_* + \|\mathbf{C}^*\|_F^2 + \langle \mathbf{G}^\infty - \mathbf{G}^*, -\mathcal{A}^*(\mathbf{M}^\infty) \rangle$$
$$+ \langle \mathbf{E}^\infty - \mathbf{E}^*, -\mathcal{B}^*(\mathbf{M}^\infty) \rangle + \langle \mathbf{C}^\infty - \mathbf{C}^*, -\mathcal{N}^*(\mathbf{M}^\infty) \rangle$$
$$= \|\mathbf{G}^*\|_1 + \|\mathbf{E}^*\|_* + \|\mathbf{C}^*\|_F^2 - \langle \mathcal{A}(\mathbf{G}^\infty - \mathbf{G}^*), \mathbf{M}^\infty \rangle \qquad (29)$$
$$- \langle \mathcal{B}(\mathbf{E}^\infty - \mathbf{E}^*), \mathbf{M}^\infty \rangle - \langle \mathcal{N}(\mathbf{C}^\infty - \mathbf{C}^*), \mathbf{M}^\infty \rangle$$
$$= \|\mathbf{G}^*\|_1 + \|\mathbf{E}^*\|_* + \|\mathbf{C}^*\|_F^2 - \langle \mathcal{A}(\mathbf{G}^\infty - \mathbf{G}^*) + \mathcal{B}(\mathbf{E}^\infty - \mathbf{E}^*) + \mathcal{N}(\mathbf{C}^\infty - \mathbf{C}^*), \mathbf{M}^\infty \rangle$$
$$= \|\mathbf{G}^*\|_1 + \|\mathbf{E}^*\|_* + \|\mathbf{C}^*\|_F^2$$

since both $(\mathbf{C}^\infty, \mathbf{G}^\infty, \mathbf{E}^\infty)$ and $(\mathbf{C}^*, \mathbf{G}^*, \mathbf{E}^*)$ are feasible solutions. So we conclude that $(\mathbf{C}^\infty, \mathbf{G}^\infty, \mathbf{E}^\infty)$ is an optimal solution to (4).

Similarly we let $k = k_i - 1$ in Lemma 16 and by the definition of subgradient, we have

$$\|\mathbf{G}\|_1 \geq \|\mathbf{G}^{k_i}\|_1 + \left\langle \mathbf{G} - \mathbf{G}^{k_i}, -\beta_{k_i-1}\tau_A(\mathbf{G}^{k_i} - \mathbf{G}^{k_i-1}) - \mathcal{A}^*(\bar{\mathbf{M}}^{k_i}) \right\rangle \qquad (30)$$

for any $\mathbf{G}$. Fix $\mathbf{G}$ and let $i \to \infty$, we see that

$$\|\mathbf{G}\|_1 \geq \|\mathbf{G}^\infty\|_1 + \langle \mathbf{G} - \mathbf{G}^\infty, -\mathcal{A}^*(\mathbf{M}^\infty) \rangle$$

for any $\mathbf{G}$. So $-\mathcal{A}^*(\mathbf{M}^\infty) \in \partial\|\mathbf{G}^\infty\|_1$. Similarly, $-\mathcal{B}^*(\mathbf{M}^\infty) \in \partial\|\mathbf{E}^\infty\|_*$. It is also not difficult to check that $-\mathcal{N}^*(\mathbf{M}^\infty) = \mathbf{C}$. Therefore, $(\mathbf{C}^\infty, \mathbf{G}^\infty, \mathbf{E}^\infty, \mathbf{M}^\infty)$ is a KKT point of problem (4).

Next we prove that the whole sequence of $\{(\mathbf{C}^k, \mathbf{E}^k, \mathbf{G}^k, \mathbf{M}^k)\}$ converges to $\{(\mathbf{C}^\infty, \mathbf{E}^\infty, \mathbf{G}^\infty, \mathbf{M}^\infty)\}$.

By choosing $(\mathbf{C}^*, \mathbf{G}^*, \mathbf{E}^*, \mathbf{M}^*) = (\mathbf{C}^\infty, \mathbf{G}^\infty, \mathbf{E}^\infty, \mathbf{M}^\infty)$ in Lemma 17, we have $\tau_A\|\mathbf{G}^{k_i} - \mathbf{G}^\infty\|_F^2 + \tau_B\|\mathbf{G}^k - \mathbf{G}^\infty\|_F^2 + \beta_{k_i}^{-2}\|\mathbf{M}^{k_i} - \mathbf{M}^\infty\|_F^2 \to 0$. By Lemma 17, we readily have $\tau_A\|\mathbf{G}^k - \mathbf{G}^\infty\|_F^2 - \|\mathcal{A}(\mathbf{G}^k - \mathbf{G}^\infty)\|_F^2 + \tau_B\|\mathbf{M}^k - \mathbf{M}^\infty\|_F^2 + \beta_k^{-2}\|\mathbf{M}^k - \mathbf{M}^\infty\|_F^2 \to 0$. So $(\mathbf{C}^k, \mathbf{G}^k, \mathbf{E}^k, \mathbf{M}^k) \to (\mathbf{C}^\infty, \mathbf{G}^\infty, \mathbf{E}^\infty, \mathbf{M}^\infty)$. Since $(\mathbf{C}^\infty, \mathbf{G}^\infty, \mathbf{E}^\infty, \mathbf{M}^\infty)$ can be an arbitrary accumulation point of $(\mathbf{C}^k, \mathbf{G}^k, \mathbf{E}^k, \mathbf{M}^k)$, we can conclude that $(\mathbf{C}^k, \mathbf{G}^k, \mathbf{E}^k, \mathbf{M}^k)$ converges to a KKT point. Since KKT point is the global optimal solution in the convex problem, $(\mathbf{C}^k, \mathbf{G}^k, \mathbf{E}^k, \mathbf{M}^k)$ converges to a global minimizer. ∎

4 Algorithm

In this section we establish the derivation for the closed-form solution of each subproblem. The four steps are noted as Updating $\mathbf{C}$, Updating $\mathbf{E}$, Updating $\mathbf{G}$ and Updating $\mathbf{M}$.

Updating $\mathbf{C}$:

$$\mathbf{C}^{k+1} = \arg\min_{\mathbf{C}} \frac{1}{2}\|\mathbf{C}\|_F^2 + \left\langle \mathbf{M}_2^k, \mathbf{E}^k - \mathbf{X}^T\mathbf{G}^k\mathbf{Y} - \mathbf{C} \right\rangle + \frac{\beta_k}{2}\|\mathbf{E}^k - \mathbf{X}^T\mathbf{G}^k\mathbf{Y} - \mathbf{C}\|_F^2 \qquad (31)$$

which has a closed form solution as:

$$\mathbf{C}^{k+1} = \frac{\beta_k}{\beta_k + 1}(\mathbf{E}^k - \mathbf{X}^T\mathbf{G}^k\mathbf{Y} + \mathbf{M}_2^k/\beta_k) \qquad (32)$$

Updating $\mathbf{G}$:

$$\min_{\mathbf{G}} \lambda_G\|\mathbf{G}\|_1 + \left\langle \mathbf{M}_2, \mathbf{E}^k - \mathbf{X}^T\mathbf{G}\mathbf{Y} - \mathbf{C}^k \right\rangle + \frac{\beta_k}{2}\|\mathbf{E}^k - \mathbf{X}^T\mathbf{G}\mathbf{Y} - \mathbf{C}^k\|_F^2, \qquad (33)$$

after adding constant term to Eq. (33) we obtain

$$\min_{\mathbf{G}} \lambda_G\|\mathbf{G}\|_1 + \frac{\beta_k}{2}\|\mathbf{B}^k - \mathbf{X}^T\mathbf{G}\mathbf{Y} - \mathbf{C}^k\|_F^2 \qquad (34)$$

where $\mathbf{B}_1^k = \mathbf{E}^k + \mathbf{M}_2^k/\beta_k$. By converting the matrix $\mathbf{b}$ into a vector $\mathbf{g} = \text{vec}(\mathbf{G})$, $\text{vec}(\mathbf{X}^T\mathbf{G}\mathbf{Y}) = (\mathbf{Y}^T \otimes \mathbf{X}^T)\mathbf{g}$. Further we let $\mathbf{b}^k = \text{vec}(\mathbf{B}^k)$ and $\otimes$ computes the Kronecker product of two matrices. Thus, if we denote $\mathbf{A} = (\mathbf{Y}^T \otimes \mathbf{X}^T)$, the above subproblem becomes:

$$\min_{\mathbf{g}} \lambda_G\|\mathbf{g}\|_1 + \frac{\beta_k}{2}\|\mathbf{A}\mathbf{g} + \mathbf{c}^k - \mathbf{b}_1^k\|_2^2 \qquad (35)$$

Since (35) is a lasso problem, which does not have a closed-form solution and must be solved iteratively in practice, by utilizing a linearization technique, we have

$$\frac{1}{2}\|\mathbf{A}\mathbf{g} + \mathbf{c}^k - \mathbf{b}_1^k\|_2^2 \approx \frac{1}{2}\|\mathbf{A}\mathbf{g}^k + \mathbf{c}^k - \mathbf{b}_1^k\|_2^2 + \langle f_1^k, \mathbf{g} - \mathbf{g}^k \rangle + \frac{\tau_A}{2}\|\mathbf{g} - \mathbf{g}^k\|_2^2 \tag{36}$$

where $\tau_A > 0$ is a proximal parameter and

$$f_1^k = \mathbf{A}^T(\mathbf{A}\mathbf{g}^k + \mathbf{c}^k - \mathbf{b}_1^k) = \mathbf{A}^T(\mathbf{A}\mathbf{g}^k + \mathbf{c}^k - \mathbf{e}^k - \mathbf{m}_2^k/\beta_k) \tag{37}$$

is the gradient of $\frac{1}{2}\|\mathbf{A}\mathbf{g} + \mathbf{c}^k - \mathbf{b}_1^k\|_2^2$ at $\mathbf{g}_k$. Eq. (20) can be re-written as:

$$\min_{\mathbf{g}} \lambda_G\|\mathbf{g}\|_1 + \frac{\beta_k \tau_A}{2}\|\mathbf{g} - [\mathbf{g}^k - f_1^k/\tau_A]\|_2^2 \tag{38}$$

Obviously the closed-form solution is:

$$\mathbf{g}^{k+1} = \max(|\mathbf{g}^k - f_1^k/\tau_A| - \frac{\lambda_G}{\tau_A \beta_k}, 0) \odot sgn(\mathbf{g}^k - f_1^k/\tau_A) \tag{39}$$

Updating $\mathbf{E}$:

$$\min_{\mathbf{E}} \lambda_E\|\mathbf{E}\|_* + \left\langle \mathbf{M}_1^k, R_\Omega(\mathbf{E} - \mathbf{F}) \right\rangle + \frac{\beta_k}{2}\|R_\Omega(\mathbf{E} - \mathbf{F})\|_F^2$$
$$+ \left\langle \mathbf{M}_2^k, \mathbf{E} - \mathbf{X}^T\mathbf{G}^{k+1}\mathbf{Y} - \mathbf{C}^k \right\rangle + \frac{\beta_k}{2}\|\mathbf{E} - \mathbf{X}^T\mathbf{G}^{k+1}\mathbf{Y} - \mathbf{C}^k\|_F^2 \tag{40}$$

which we can reformulate as:

$$\min_{\mathbf{E}} \lambda_E\|\mathbf{E}\|_* + \frac{\beta_k}{2}\|R_\Omega(\mathbf{E} - \mathbf{B}_2^k)\|_F^2 + \frac{\beta_k}{2}\|\mathbf{E} - \mathbf{B}_3^k\|_F^2 \tag{41}$$

where $\mathbf{B}_2^k = R_\Omega(\mathbf{F} - \mathbf{M}_1^k/\beta_k)$ and $\mathbf{B}_3^k = \mathbf{X}^T\mathbf{G}^{k+1}\mathbf{Y} + \mathbf{C}^k - \mathbf{M}_2^k/\beta_k$. After linearization, the problem can be approximately optimized by:

$$\min_{\mathbf{E}} \lambda_E\|\mathbf{E}\|_* + \frac{\beta_k \tau_B}{2}\|\mathbf{E} - (\mathbf{E}^k - f_2^k/\tau_B)\|_F^2 + \frac{\beta_k \tau_B}{2}\|\mathbf{E} - (\mathbf{E}^k - f_3^k/\tau_B)\|_F^2 \tag{42}$$

where f_2^k and f_3^k are the gradients of $\frac{1}{2}\|R_\Omega(\mathbf{E} - \mathbf{B}_2^k)\|_F^2$ and $\frac{1}{2}\|\mathbf{E} - \mathbf{B}_3^k\|_F^2$ at $\mathbf{E}^k$, which are illustrated below:

$$f_2^k = R_\Omega(\mathbf{E}^k - \mathbf{B}_2^k) = R_\Omega(\mathbf{E}^k - \mathbf{F} + \mathbf{M}_1^k/\beta_k),$$
$$f_3^k = \mathbf{E}^k - \mathbf{B}_3^k = \mathbf{E}^k - \mathbf{X}^T\mathbf{G}^{k+1}\mathbf{Y} - \mathbf{C}^k + \mathbf{M}_2^k/\beta_k. \tag{43}$$

The closed-form solution is then readily obtainable as

$$\mathbf{E}^{k+1} = SVT(\mathbf{E}^k - (f_2^k + f_3^k)/(2\tau_B), \lambda_E/2(\beta_k \tau_B)) \tag{44}$$

Here the operator $SVT(\mathbf{E}, t)$ is defined in [2] for soft-thresholding the singular values of an arbitrary matrix $\mathbf{E}$ by t.

Updating $\mathbf{M}$:

$$\mathbf{M}_1^{k+1} = \mathbf{M}_1^k + \beta_k(R_\Omega(\mathbf{E}^{k+1} - \mathbf{F})),$$
$$\mathbf{M}_2^{k+1} = \mathbf{M}_2^k + \beta_k(\mathbf{E}^{k+1} - \mathbf{X}^T\mathbf{G}^{k+1}\mathbf{Y} - \mathbf{C}^{k+1}). \tag{45}$$

5 Feature Description Table of Drug Discovery Dataset

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Table 1: Drug corresponding feature.

Label	Feature Name
F1	Molecular Weight(g/mol)
F2	XLogP3
F3	Hydrogen Bond Donor Count
F4	Hydrogen Bond Acceptor Count
F5	Rotatable Bond Count
F6	Exact Mass(g/mol)
F7	Monoisotopic Mass(g/mol)
F8	Topological Polar Surface Area
F9	Heavy Atom Count
F10	Complexity
F11	Defined Atom Stereocenter Count
F12	Undefined Atom Stereocenter Count
F13	Defined Bond Stereocenter Count
F14	Covalently-Bonded Unit Count
F15	Universal

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